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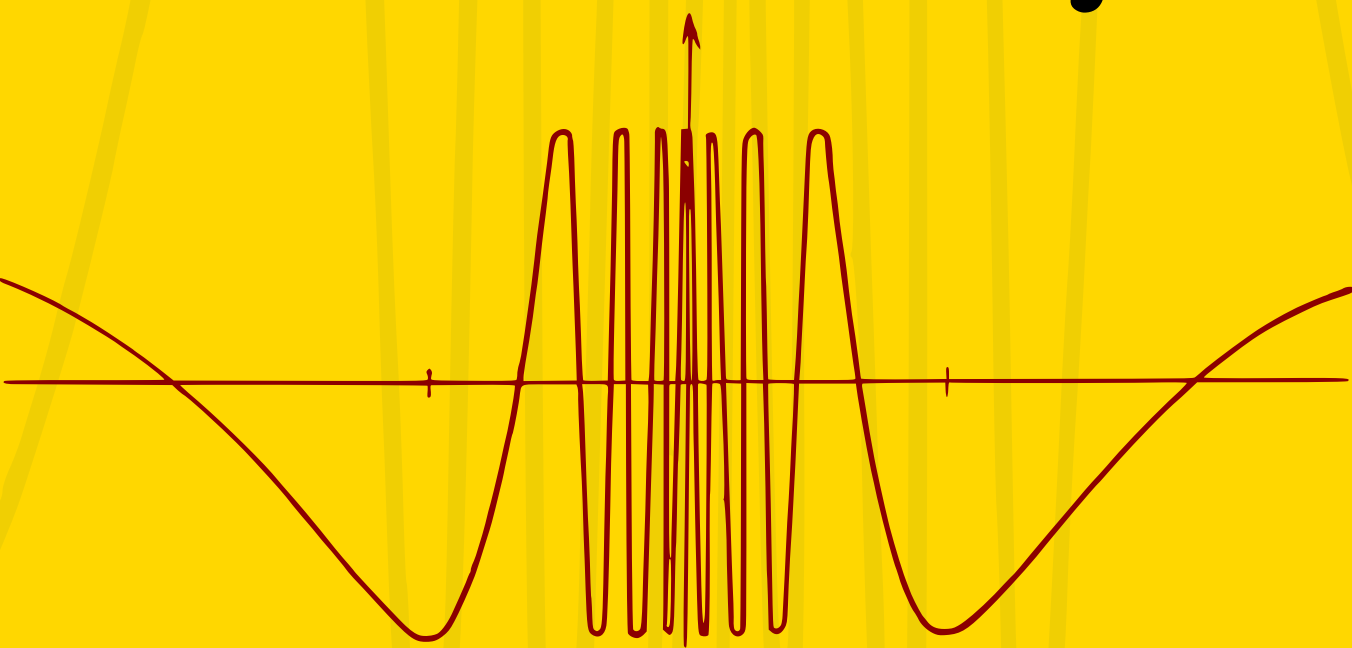
P.P. Korovkin

Limits *and* Continuity



LIMITS AND CONTINUITY

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LIMITS AND CONTINUITY

BY

P. P. KOROVKIN

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Preface

The present volume in The Pocket Mathematical Library stems in large part from Chapters 1–4 of P. P. Korovkin's *Mathematical Analysis*, Moscow (1963). The material has been heavily rewritten and supplemented by 21 problem sets, one after each section. The result is a succinct but remarkably complete introduction to the theory of limits and continuity. The book may also be thought of as a "precalculus" text in that it deals with those properties of functions which can be successfully discussed short of introducing the notion of a derivative.

Answers to the even-numbered problems will be found at the end of the book.

Contents

CHAPTER 1

Functions

1. Variables and Functions. Intervals and Sequences

For the most part, elementary mathematics deals with quantitative aspects of the real world. The quantities encountered in nature can be classified into two categories, *constants* and *variables*. (Of course, we are stretching things a bit in making this rather sweeping statement!) For example, the ratio of the circumference of a circle to its diameter is a constant, and so is the temperature at which water boils, at least "under standard conditions." Many more examples of this sort come to mind at once. However, it is fair to say that change and changeability are the rule rather than the exception. People and things move about, bridges and sidewalks expand and contract depending on time of day and season of year, the boiling point of water depends on atmospheric pressure, and so on. There is no end to a recitation of variable aspects of natural phenomena. In fact, something that has one value under certain circumstances, like the boiling point of water, may have a different value under other circumstances (witness the pressure cooker). This suggests defining a constant as a quantity which takes only one value and a variable as a quantity which takes two or more distinct values, always in the context of a given problem.

Continuing in this vein, we note that not all variables are of equal interest, and in fact the ones of greatest interest, at least to scientists, are those which depend in a recognizable way on other variables, thereby exhibiting a certain regularity of behavior. For example, the position of a missile depends on the elapsed time since firing, the boiling point of water depends on atmospheric pressure, which in turn depends on the height above sea level, the pressure of a confined gas depends both on the volume of the containing vessel and on the pressure, and so on. In other words, many natural phenomena can be successfully described in terms of two or more "related variables," where knowledge of all but one of the variables specifies the remaining variable. The variables whose values can be chosen arbitrarily, within certain limits, are called *independent variables* or arguments, while the remaining variable, which depends on these arguments, is called the *dependent variable* or *function*. These definitions are tentative, and will be made more precise later.

Example 1. The formula

$$V = lwh$$

gives the volume of a rectangular parallelepiped (think of a brick) in terms of its length l, width w and height h. In this formula, the volume V is singled out as the function. But if we write

$$h = \frac{V}{lw},$$

the height h becomes the function and the volume is now an independent variable (along with the length and width).

The example just given involves four "related variables," one dependent and three independent, or equivalently functions of three variables. However, there is a great deal to be said about the particularly simple case of only two "related variables," one dependent and one independent, i.e., about functions of one variable. In keeping with its elementary character, this book is concerned exclusively with topics involving functions of one variable.

The above remarks were intentionally of a preliminary character, and it will therefore come as no surprise that the notion of a function, as just defined, is both too general and too vague for our purposes. The reader may already have noticed a certain ambiguity attaching to the word "dependent" in the definition of a function as a "dependent variable," as revealed by the following example:

Example 2. The size of a harvested crop depends on the rainfall during the growing season. But even if we know the total amount of rainfall since planting, we cannot say anything about the crop size, without further information about how the rainfall occurred. If the rainfall all took place on the same day, followed by drought for the rest of the growing season, then we can expect a negligibly small harvest. On the other hand, if the same total amount of precipitation occurred in a pattern of alternating rainy and sunny days, then a good harvest can be expected.

This example shows that the dependence of one variable on another, like crop size on rainfall, may involve factors other than those explicitly stated ("hidden variables," as it were), and that this possibility will have to be precluded from the outset if our definition of a function is to be at all useful. Moreover, while we are at it, it would be convenient to include the possibility of a function taking the same value for all values of its arguments, which is tantamount to regarding a constant as a special case of a variable. Finally, as shown by our next two examples, a suitable definition of a function must also involve an explicit statement about the permissible values of the independent variable.

Example 3. Suppose an object is dropped from a height of 19.6 meters. Then, as we know from elementary physics, the motion of the object is described quite accurately (for a while, at least) by the formula

$$s = \tfrac{1}{2}gt^2, \tag{1.1}$$

where g is the acceleration of gravity (9.8 meters per second), s is the distance fallen, and t is the elapsed time since the object was released. But this formula does not describe the motion of the object for all values of t. In fact, after falling for two seconds, the object strikes the ground and is subsequently motionless. In other words, formula (1.1) is valid only during the interval $0 \leqslant t \leqslant 2$ seconds, and the subsequent (rather trivial!) behavior of the object is described by the formula

$$s = 19.6 \text{ meters.}$$

Incidentally, this shows the desirability of considering "constant functions."

Example 4. Consider the same example, but suppose that now the object is dropped from a height of 176.4 meters. Then the function characterizing the motion of the falling body is again given by formula (1.1), which is now valid during the longer interval $0 \leqslant t \leqslant 6$.

We are now in a position to give a formal definition of a function, after imposing one last condition, namely that the values of both the independent and the dependent variables be *real numbers.* Here we assume that the reader is familiar with the real number system (some properties of which will be discussed in Sec. 5), in particular the fact that there is a one-to-one correspondence[1] between the real numbers and the points of a line, hence called the "real line" (after introducing an origin and a unit of length).

Thus let X be a set of real numbers, or equivalently a set of points on the real line. Then by a *function defined on* (or *in*) X we mean a rule associating a real number y with every element x of the set X. Symbolically, this is indicated by writing.

$$y = f(x) \text{ if } x \in X,$$

or simply

$$y = f(x),\ x \in X,$$

1. A correspondence between two sets X and Y is called *one-to-one* if one and only one element y in Y corresponds to each element x in X, and if one and only one element x in X corresponds to each element y in Y.

where by $x \in X$ we mean that the element x belongs to the set X. The set X is called the *domain (of definition)* of the function, and the set Y of all y such that $y = f(x)$ for some $x \in X$ is called the *range* of the function. The range Y can always be deduced from a knowledge of the domain X (which is an intrinsic part of the definition of the function) and the rule associating y with x. The number $y = f(x)$ is called the *value* of the function at the point x (geometric language is customary here). Logically, there is a distinction between the function itself, often denoted by just f, and its value at a given point x, denoted by $f(x)$. In fact, the function itself is the set of all ordered pairs of the form $(x, f(x))$, where $x \in X$. However, having called attention to this distinction, we henceforth proceed to use the symbol $f(x)$ for both the function and its values.

As already noted, unless the contrary is explicitly stated, the domain of definition of a function will always be a set of real numbers, often the set of *all* real numbers. In this regard, sets of the following kinds are particularly important:

1. The set of all x such that $a < x < b$, called an *open interval* and denoted by (a, b);
2. The set of all x such that $a \leqslant x \leqslant b$, called a *closed interval*, and denoted by $[a, b]$;

The seemingly slight difference between open and closed intervals (closed intervals contain their end points, but open intervals do not) is actually crucial. In fact, it turns out that there are many results valid for functions defined in closed intervals which do not apply to functions defined in open intervals. In the case of open intervals, we permit the values $a = -\infty$ and $b = +\infty$. Specifically, by $(-\infty, b)$ we mean the set of all x less than b, by $(a, +\infty)$ the set of all x greater than a, and by $(-\infty, +\infty)$ the set of all x, i.e., the whole real line. Thus open intervals may be infinitely long, but closed intervals must be of finite length (since $-\infty$ and $+\infty$ are not regarded as real numbers).

It is also important to consider intervals where one end point is included and the other omitted. Such intervals are half-open or half-closed, depending on one's point of view. Thus the set of all x such that $a < x \leqslant b$ is called a *left half-open* (or *right half-closed*) interval, and is denoted by $(a, b]$. Similarly, the set of all x such that $a \leqslant x \leqslant b$ is called a *right half-open* (or *left half-closed*) interval, and is denoted by $[a, b)$.

Remark. In many cases, the context makes it clear whether we are dealing with open or closed intervals (or half-open intervals). We then drop the qualifying adjectives, and simply talk about *intervals.*

Obviously, functions can be defined on domains other than intervals. A particularly important example is that of a function $f(x)$ defined on the set $X = \{1, 2, \ldots, n, \ldots\}$ of all positive integers. Such a function is called an *(infinite) sequence*, and the argument is usually written as a subscript, e.g.,

$$y_n = f(n).$$

The sequence itself is usually denoted by $y_1, y_2, \ldots, y_n, \ldots$, or simply by $\{y_n\}$. The number y_1 is called the *first term* of the sequence, y_2 the *second term*, ... , y_n the *n'th term*, and so on. The number y_n is also called the *general term* of the sequence $\{y_n\}$, since as n runs through all the positive integers, y_n runs through the whole sequence $\{y_n\}$.

To recapitulate, a function is a rule associating a real number y with every element x of a given set of real numbers X.[2] Thus the sentence

> "With every number in the interval [1, 4], associate the square of the number plus three times the number"

specifies a function, whose value at the point 2, say, is $2^2 + 3 \cdot 2 = 10$. This example is particularly simple, and one can easily imagine functions whose specification would require a great many more words. To avoid such complications and make things as clear as possible, we shall often specify functions

2. The symbols x, y and X are historically favored in statements like this, but there is nothing sacred about them, and other symbols would do as well.

symbolically. One symbolic way of writing the function just described in words is

$$y = x^2 + 3x, \quad x \in [1, 4].$$

Other, less concise ways are

$$y = x(x + 3), \quad x \in [1, 4],$$

$$y = (x + 1)^2 + x - 1, \quad x \in [1, 4].$$

Example 5. The formula

$$y = f(x) = \begin{cases} x + 3, & x \in [0, 1], \\ x^2, & x \in (1, 4] \end{cases}$$

defines a function in the interval [0, 4]. Thus for $x = \frac{1}{2}$ we have

$$y = f(\tfrac{1}{2}) = \tfrac{1}{2} + 3 = \tfrac{7}{2},$$

while for $x = 3$,

$$y = f(3) = 3^2 = 9.$$

It is important to note that although the rule takes different symbolic forms in the subintervals [0, 1] and (1, 4], there is still only a single rule assigning a real number y to every real number in the interval [0, 4].

Example 6. Similarly, the formula

$$y = f(x) = \begin{cases} x - 1, & x \in [0, 2], \\ x^2 - 3, & x \in (2, 5), \\ 11x, & x \in [5, 6], \\ x^3 - 150, & x \in (6, 9] \end{cases}$$

defines a function in the interval [0, 9], although the rule again takes a different form in each of the (four) subintervals.

Example 7. The functions

$$y = x^3, \quad x \in (-\infty, +\infty)$$

and

$$y = x^3 \cos^2 x + x^3 \sin^2 x, \quad x \in (-\infty, +\infty)$$

are identical, since they have the same domain $(-\infty, +\infty)$ and coincide at every point of this domain.

Example 8. The functions

$$y = f(x) = x^2, \quad x \in [0, 1]$$

and

$$y = g(x) = x^2, \quad x \in [0, 4]$$

are different, since they have different domains. Thus $f(2)$ is meaningless, whereas $g(2) = 2^2 = 4$.

More insight into the generality of the function concept is provided by the following function, which will play an important role in the construction of counterexamples:

DEFINITION. *The function*

$$D(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is called the Dirichlet function.

PROBLEMS

1. If

$$f(x) = 2x^3 - 3x + 4$$

for all real x, find $f(-2)$, $f(0)$, $f(1)$ and $f(\sqrt{3})$.

2. Consider the function

$$\varphi(x) = \frac{2x - 1}{3x^2 - 1}, \quad x \in (-1, 1).$$

Do $\varphi(1/\sqrt{3})$, $\varphi(1/\sqrt{2})$ and $\varphi(1)$ exist?

3. Given that

$$\psi(x) = \begin{cases} 2^x & x \in (-1, 0), \\ 2, & x \in [0, 1), \\ x - 1, & x \in [1, 0], \end{cases}$$

find $\psi(2)$, $\psi(0)$, $\psi(0.5)$, $\psi(-0.5)$ and $\psi(3)$.

4. Write the first 5 terms of the sequence $\{y_n\}$ with general term

a) $y_n = n^2$;

b) $y_n = \dfrac{1}{n(n+1)}$;

c) $y_n = (-1)^{n-1} \dfrac{1}{n}$;

d) $y_n = 2^n$.

5. Consider the sequence $\{y_n\}$ such that

$$y_n = \begin{cases} 1 & \text{if } n = 1, 2, \\ y_{n-1} + y_{n-2} & \text{if } n > 2. \end{cases}$$

Write the first 8 terms of $\{y_n\}$.

Comment. The terms of $\{y_n\}$ are called the *Fibonacci numbers*.

6. Let $D(x)$ be the Dirichlet function. Prove that

$$D^n(x) = D(x)$$

for every positive integer n.

7. Suppose

$$\chi(x) = \begin{cases} 1 \text{ if } x \text{ is rational,} \\ -1 \text{ if } x \text{ is irrational.} \end{cases}$$

Prove that

$$\chi^n(x) = \chi(x)$$

for every odd integer n. How about the case of even n?

2. Absolute Values. Neighborhoods

A particularly important function is the *absolute value* of x, defined by

$$|x| = \begin{cases} x \text{ if } x \geqslant 0, \\ -x \text{ if } x < 0. \end{cases}$$

Since

$$\sqrt{x^2} = \begin{cases} x \text{ if } x \geqslant 0, \\ -x \text{ if } x < 0, \end{cases}$$

we can also write

$$|x| = \sqrt{x^2}$$

for all real x.

It follows that

$$|xy| = \sqrt{(xy)^2} = \sqrt{x^2 y^2} = \sqrt{x^2}\,\sqrt{y^2} = |x|\,|y|,$$

$$\left|\frac{x}{y}\right| = \sqrt{\left(\frac{x}{y}\right)^2} = \sqrt{\frac{x^2}{y^2}} = \frac{\sqrt{x^2}}{\sqrt{y^2}} = \frac{|x|}{|y|} \quad (y \neq 0).$$

Thus the absolute value of a product or quotient is the product or quotient of the absolute values. However, for sums and differences we can only write inequalities. In fact,

$$|x + y| = \sqrt{(x + y)^2} = \sqrt{x^2 + 2xy + y^2}$$

$$\leqslant \sqrt{x^2 + 2|x|\,|y| + y^2} = \sqrt{(|x| + |y|)^2} = |x| + |y|,$$

i.e.,

$$|x + y| \leqslant |x| + |y|, \tag{1.2}$$

while

$$|x| = |(x - y) + y| \leqslant |x - y| + |y|,$$

and hence

$$|x - y| \geqslant |x| - |y|. \tag{1.3}$$

The absolute value of the difference of two numbers has a simple geometric interpretation. Let A be the point with coordinate x_1 and B the point with coordinate x_2. Then the length of the line segment AB, denoted by $\overline{AB}$, equals $x_1 - x_2$ if $x_1 > x_2$ and $x_2 - x_1$ if $x_2 > x_1$, which means that

$$\overline{AB} = |x_1 - x_2|$$

regardless of the relative positions of x_1 and x_2. In other words, the length of an interval equals the absolute value of the difference between its end points, taken in either order. The set of points x satisfying the inequality

$$|x - a| < \varepsilon \tag{1.4}$$

is just the set of points whose distance from the point a is less than ε, i.e., the open interval $(a - \varepsilon, a + \varepsilon)$. Sets of the form (1.4) are important enough to warrant a special

DEFINITION. *An open interval with midpoint a is called a neighborhood of a.*

PROBLEMS

1. Prove the inequalities (1.2) and (1.3) geometrically.

2. If $f(x) = x + 1$ and $\varphi(x) = x - 2$, solve the equation

$$|f(x) + \varphi(x)| = |f(x)| + |\varphi(x)|.$$

3. Prove the identity

$$\left(\frac{x + |x|}{2}\right)^2 + \left(\frac{x - |x|}{2}\right) = x^2.$$

4. Find all x such that

a) $|x + 1| < 0.01$; b) $|x - 2| \geqslant 10$; c) $|x| > |x + 1|$;

d) $|2x - 1| < |x - 1|$; e) $|x + 2| + |x - 2| \leqslant 12$;

f) $|x + 2| - |x| > 1$; g) $||x + 1| - |x - 1|| < 1$.

5. Find nonoverlapping neighborhoods of the points $-1, 0, \frac{1}{2}$.

6. By *signum* of x is meant the function

$$\operatorname{sgn} x = \begin{cases} 1 \text{ if } x > 0, \\ 0 \text{ if } x = 0, \\ -1 \text{ if } x < 0. \end{cases}$$

Prove that $|x| = x \operatorname{sgn} x$.

3. Graphs and Tables

By the *graph* of a function $y = f(x)$ defined on a set X, we mean the set of all points in the xy-plane of the form $(x, f(x))$, where $x \in X$. Since in most cases of interest, X contains infinitely many points, the practical construction of graphs usually involves some kind of approximation. In fact, the customary procedure, familiar from elementary mathematics, is to plot a number of "typical points" $(x, f(x))$ and then join them by a "smooth curve." If this procedure is properly executed, the graph becomes more accurate, i.e., "closer to the true function $f(x)$," as more points are plotted. [Of course, this presupposes sufficient "regularity" on the part of $f(x)$.] Graphs can often reveal important properties of a function at a glance.

Example 1. The function with the graph shown in Figure 1 takes its largest value for $x = x_1$ and its smallest value for $x = x_2$ The intervals in which the function increases and decreases can be read off directly from the graph.

Example 2. The Dirichlet function $D(x)$ defined on p. 8 is so "irregular" that it cannot be plotted at all! Obviously, there are infinitely many points of the x-axis where $D(x) = 0$, and also infinitely many points where $D(x) = 1$. Moreover, between any two points where $D(x) = 0$, there is a point where $D(x) = 1$, and *vice versa.* In other words, the points $(x, D(x))$ are "everywhere dense" in the parallel lines $y = 0$ and $y = 1$, and any attempt to join points $(x, D(x))$, no matter how close together, will always involve an error "of order unity."

In many cases, one does not plot a graph starting from an explicit formula $y = f(x)$. Instead one is presented directly with a graph representing the readings of some scientific instrument (e.g., a seismograph or altimeter) and showing how a certain variable depends on another (usually the time). This, then, is another method for specifying functions, distinct from those considered in Sec. 1. However, it should be kept in mind that such graphs always involve intrinsic inaccuracies, due to experimental errors, not to mention the impossibility (and irrelevance) of trying to read the graphs to arbitrarily high accuracy. Thus we shall use graphs mainly as a visual aid, to help study the behavior of functions known accurately in advance.

A function can also be specified by means of a *table*, listing values of the function for various values of the argument. Tables may be used as a convenience to specify functions given by explicit formulas, e.g., trigonometric functions, exponentials and logarithms, etc. However, one may also be presented directly with a table giving the output of some scientific instrument, e.g., a digital computer. Clearly, graphs and tables are related concepts. For example, the process of joining plotted points by a smooth curve is the graphical equivalent of interpolating between values of a function listed in a table.

PROBLEMS

1. Construct the graphs of the following functions:

a) $y = 2x - 3$; b) $y = x^2 + 1$; c) $y = x^3$; d) $y = \dfrac{1}{x}$;

e) $y = \dfrac{1}{x - 1}$; f) $y = \dfrac{1}{x^2 - 1}$;

g) $y = \begin{cases} x^2 \text{ if } -2 \leqslant x < 0, \\ 0 \text{ if } x = 0, \\ 2x - 1 \text{ if } 0 < x \leqslant 2, \end{cases}$

h) $y = |x|$; i) $y = x - |x|$.

2. Solve the equation

$$x^3 - 4x - 1 = 0$$

approximately by drawing a graph of the function $y = x^3 - 4x - 1 = 0$.

3. Express the radius R of a sphere as a function of its volume V, and draw the graph of this function.

4. The function given by the following table is familiar from everyday life:

x	y
0	32
20	68
	104
60	140
80	
100	212

What is it? Fill in the missing entries in the table. Find a formula relating y to x and one relating x to y.

4. Some Simple Function Classes

In this book, we shall time and again deal with functions specified by formulas. Unless the contrary is explicitly stated, the domain of such a function will be the set of all (real) values of x for which the operations called for in the given formula can actually be carried out. In particular we shall always insist that the result of the operations be real and finite, like x itself.

Example 1. The function $\sqrt{x}$ exists for all nonnegative x, since it is for just such x that $\sqrt{x}$ is real.

Example 2. The function $1/x$ exists for all nonzero x, since it is for just such x that $1/x$ is finite.

Among the simplest functions of x are the powers $x, x^2, x^3, \ldots$ The even powers $x^2, x^4, x^6, \ldots$ have the property of remaining the same when the sign of the argument is changed, i.e.,

$$(-x)^{2n} = x^{2n},$$

and other functions have this property, for example $\cos(-x) = \cos x$. On the other hand, the odd powers $x, x^3, x^5, \ldots$ have the property of changing sign when the sign of the argument is changed, i.e.,

$$(-x)^{2n+1} = -x^{2n+1},$$

and again other functions have the same property, for example $\sin(-x) = -\sin x$. These considerations suggest the following

DEFINITION 1. *A function $f(x)$ is said to be even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$.*

Clearly, it is tacitly assumed that the domain of an even or odd function is symmetric with respect to the origin. Moreover, the graph of an even function is symmetric with respect to the y-axis, while that of an odd function is symmetric with respect to the origin. This fact is helpful in constructing graphs of even and odd functions, since once we know the graph to the right of the y-axis, reflection can be used to find the graph to the left of the y-axis. Figure 2a shows the graphs of the even functions x^2, x^4, x^6, while Figure 2b shows the graphs of the odd functions x, x^3, x^5.

Obviously, there are functions which are neither even nor odd, e.g., $x^2 + x$. However, $x^2 + x$ is the sum of an even and

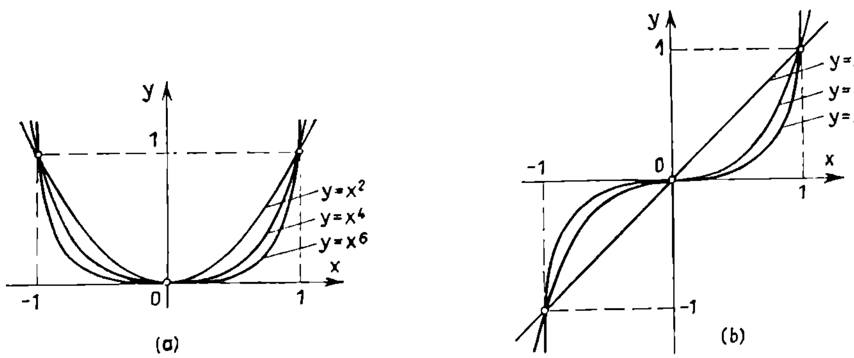

Fig. 2

an odd function, and one might ask whether this is true in general. The answer is given by

THEOREM 1.1. *A function $f(x)$ can be represented as the sum of an even function and an odd function if and only if the domain of $f(x)$ is symmetric with respect to the origin.*

Proof. There is obviously no such representation if the domain of $f(x)$ is asymmetric, as in the case of the function $\log(x + 1)$, defined only for $x > -1$. If the domain is symmetric, the desired representation is

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2},$$

where the first term in the sum is even and the second odd.

Next we introduce another important class of functions:

DEFINITION 2. *A function $f(x)$ with domain X is said to be periodic, with period $\omega \neq 0$,* if *$x \in X$ implies $x \pm \omega \in X$ and if*

$$f(x + \omega) = f(x)$$

for all $x \in X$.

THEOREM 1.2. *If $f(x)$ is periodic with period ω, then*

$$f(x + n\omega) = f(x) \quad (n = 0, \pm 1, \pm 2, \ldots) \tag{1.5}$$

for all $x \in X$.

Proof. There is nothing to prove if $n = 0$ or 1. The relation (1.5) holds for $n = -1$, since

$$f(x) = f(x - \omega + \omega) = f(x - \omega)$$

for all $x \in X$, where $f(x - \omega)$ is defined, since $x - \omega \in X$. More generally, if $n > 0$,

$$f(x + n\omega) = f(x + (n - 1)\,\omega) = \cdots = f(x + \omega) = f(x),$$
$$f(x - n\omega) = f(x - (n - 1)\,\omega) = \cdots = f(x - \omega) = f(x)$$

for all $x \in X$.

Example 3. The trigonometric functions $\sin x$ and $\cos x$, defined for all x, are periodic with period 2π (see Fig. 3).

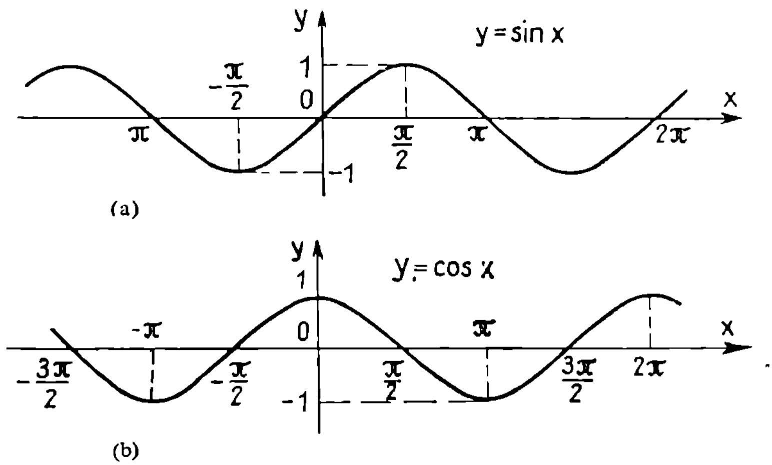
y
y = sin x
1
0
-1
x
π
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
2π
(a)
y
y = cos x
1
0
-1
x
-π
π
2π
$-\frac{3\pi}{2}$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
$\frac{3\pi}{2}$
(b)

Fig. 3

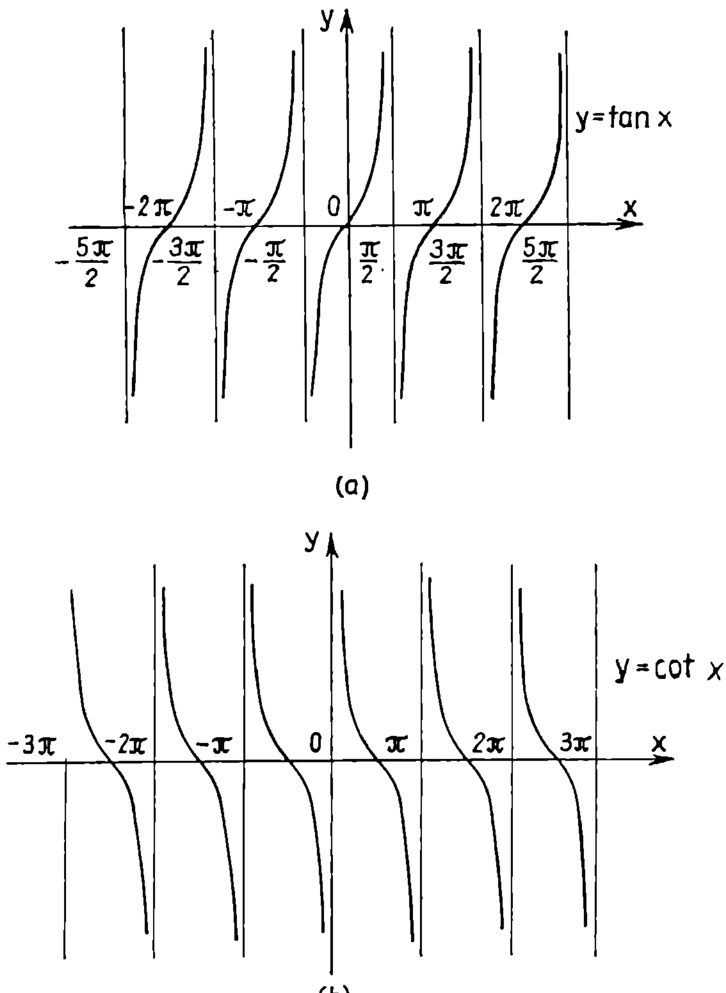
y
y = tan x
x
-2π
-π
0
π
2π
$-\frac{5\pi}{2}$
$-\frac{3\pi}{2}$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
$\frac{3\pi}{2}$
$\frac{5\pi}{2}$
(a)
y
y = cot x
x
-3π
-2π
-π
0
π
2π
3π
(b)

Fig. 4

Example 4. The trigonometric functions tan x and cot x, the first defined for all $x \neq (n + \frac{1}{2})\pi$ and the second for all $x \neq n\pi$, are periodic with period π (see Fig. 4).

Example 5. The largest integer $\leqslant x$ is called the *integral part* of x, denoted by $[x]$. Then the function

$$\{x\} = x - [x],$$

called the *fractional part* of x and defined for all x, is periodic with period 1 (see Fig. 5). This periodic function is not trigonometric.

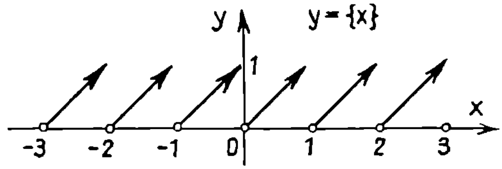

Fig. 5

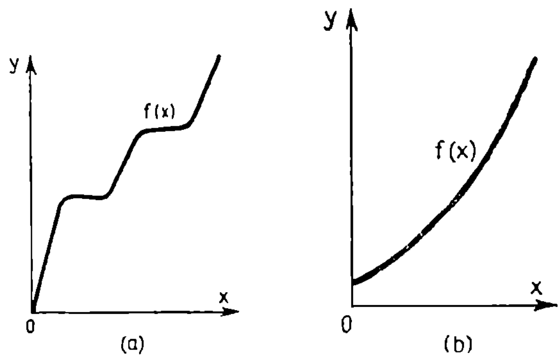

Fig. 6

Finally we introduce the class of monotonic functions:

DEFINITION 3. *A function $f(x)$ with domain X is said to be increasing (or nondecreasing) on a set $E \subset X$ if for every pair of points x* and *x' in E, $x < x'$ implies $f(x) \leqslant f(x')$.*[3] *If $x < x'$ implies $f(x) < f(x')$, then $f(x)$ is said to be strictly increasing. Decreasing (or nonincreasing) functions and strictly decreasing functions are defined similarly by writing $f(x) \geqslant f(x')$ instead of*

3. $E \subset X$ means that E is a *subset* of X, i.e., every point of E is a point of X.

$f(x) \leqslant f(x')$ and $f(x) > f(x')$ instead of $f(x) < f(x')$. A function is said to be (strictly) monotonic if it is (strictly) increasing or (strictly) decreasing.

Each of the functions shown in Figure 6 is increasing, but only the one shown in Figure 6b is strictly increasing. Similarly, each of the functions shown in Figure 7 is decreasing, but only the one shown in Figure 7b is strictly decreasing. A strictly increasing function achieves its smallest value for the smallest $x \in X$ and its largest value for the largest $x \in X$ (state the analogous property of a strictly decreasing function). A function which is increasing (decreasing) but not strictly increasing (decreasing) must have at least one interval in which it is constant, as shown in Figures 6a and 7a.

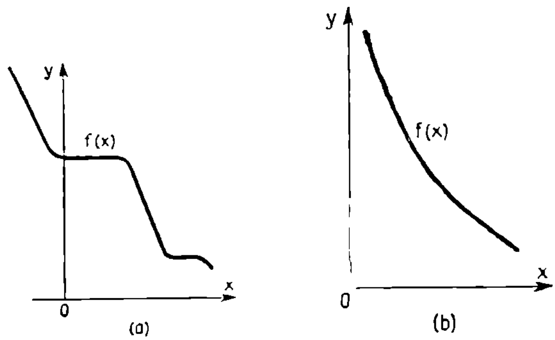

Fig. 7

PROBLEMS

1. Find the domain of each of the following functions:

a) $y = \dfrac{x}{x-1}$; b) $y = \dfrac{x}{x^2 + 2x - 3}$;

c) $y = \sqrt{2x^2 + x + 8}$; d) $y = \sqrt{4 - x^2}$; e) $y = x - |x|$;

f) $y = \dfrac{1}{\sqrt{x - |x|}}$; g) $y = \dfrac{1}{\sqrt{|x| - x}}$.

2. Which of the following functions are even:

a) $y = x^6 - 2x^4 + 4$; b) $y = x^4 - 2$; c) $y = x^5 - x$;

d) $y = 2x^3 + 3$; e) $y = 2^x$; f) $y = \sin(x - 1)$;

g) $y = \sin 2x$; h) $y = \cos(x + 1)$; i) $y = \cos 5x$;

j) $y = x^3 - 2$; k) $y = x + \sin x$; l) $y = \sqrt{1 - x^2}$;

m) $y = \log(x + \sqrt{x^2 + 1})$?

Which are odd?

3. Write each of the following functions as a sum of an even and an odd function:

a) $y = x^4 + 2x^3 + x^2 - 4$; b) $y = \sin x + \cos x - \tan x$;

c) $y = (1 + x)^6$; d) $y = \sin x + x^2 + x - 1$;

e) $y = \sin(x + \alpha)$.

4. Which of the following functions are periodic:

a) $y = \sin^2 x$; b) $y = \sin x^2$; c) $y = x \cos x$;

d) $y = \cos 2x$; e) $y = \sin \pi x$; f) $y = \sin \dfrac{1}{x}$; g) $y = 4$;

h) $y = \sin(x + 1)$; i) $y = \cos(x - 2)$?

Find the smallest positive period (if any) of each periodic function.

5. Prove that the Dirichlet function $D(x)$ defined on p. 8 is periodic, with any rational number $r > 0$ as a period.

6. Find open intervals in which each of the following functions is monotonic (or strictly monotonic):

a) $y = 2x - 1$; b) $y = x^2$; c) $y = x^2 + 2x + 5$;

d) $y = 2^x$; e) $y = x^3$; f) $y = |x|$; g) $y = |x| - x$;

h) $y = \sin x$.

5. Real Numbers and Decimal Expansions

It is already obvious that real numbers play a key role in our considerations. In a more advanced course (on modern algebra or real variable theory), you will learn how the real number system is constructed starting from the rational numbers. For our purposes, it is enough to regard the following facts about real numbers as known:

1. There is a one-to-one correspondence between the real numbers and decimal expansions of the form

$$x = \pm A.a_1a_2 \dots a_n \dots,$$

where A is zero or a positive integer and every a_i is one of the digits from 0 to 9, provided decimal expansions ending in an infinite run of nines are excluded.[4] Every real number is either rational or irrational.

2. There is a one-to-one correspondence between the rational numbers and decimal expansions like

$$\tfrac{1}{5} = 0.\underline{20}, \quad \tfrac{1}{6} = 0.1\underline{6}, \quad \tfrac{1}{7} = 0.\underline{142857}, \quad \tfrac{1}{8} = 0.125\underline{0} \dots,$$

where the underlined block of digits repeats itself indefinitely.[5]

3. There is a one-to-one correspondence between the irrational numbers and decimal expansions like

$$\sqrt{2} = 1.414213562 \dots, \quad \pi = 3.141592653 \cdots$$

which never repeat themselves.

Although familiarity makes these properties quite plausible, they are not at all trivial to prove. In fact, to do so, one cannot

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4. In other words, we write

$$\tfrac{1}{5} = 0.200000 \dots, \quad \tfrac{1}{8} = 0.125000 \dots, \text{ etc.}$$

instead of

$$\tfrac{1}{5} = 0.199999 \dots, \quad \tfrac{1}{8} = 0.124999 \dots, \text{ etc.}$$

5. Actually we can drop infinite runs of zeros, writing $\frac{1}{5} = 0.2$, $\frac{1}{8} = 0.125$, etc.

avoid going through the construction of the real number system, by some method or other. However, as already mentioned, such a construction lies beyond the scope of this book.

Next we introduce some simple but indispensable notions involving sets of real numbers:

DEFINITION 1. *A set (of real numbers) E is said to be bounded from below if there exists a number m, called a lower bound of E, such that* $m \leqslant x$ *for every* $x \in E$. *Similarly, E is said to be bounded from above if there exists a number M, called an upper bound of E, such that* $x \leqslant M$ *for every* $x \in E$. *A set is said to be bounded if it is bounded both from below and from above.*

Remark. Obviously, E is bounded if and only if E is contained in some (finite) closed interval $[a, b]$.

DEFINITION 2. *A lower bound of E which belongs to E is called the minimum of E, denoted by* min E. *An upper bound of E which belongs to E is called the maximum of E, denoted by* max E.

Remark. In other words, $\min E \leqslant x \leqslant \max E$ for all $x \in E$, and moreover $\min E \in E$, $\max E \in E$.

Example 1. Every finite set has a minimum and a maximum. Thus, if[6]

$$E = \left\{\sqrt{5}, 2, \pi, \sqrt[3]{65}, 3\right\},$$

then $\min E = 2$, $\max E = \sqrt[3]{65}$.

Example 2. The set $E = \{0, -1, 1, -2, 2, \ldots\}$ has neither a minimum nor a maximum.

Example 3. The open unit interval $E = (0, 1)$ has neither a minimum nor a maximum. In fact, if α belongs to E, then so does the still smaller number $\frac{1}{2}\alpha$ and the still larger number $\frac{1}{2}(\alpha + 1)$.

THEOREM 1.3. *If* $\lambda = \min E$, *then no lower bound of E can exceed* λ. *Similarly, if* $\Lambda = \max E$, *then no upper bound of E can be less than* Λ.

Proof. We need only note that λ and Λ belong to E.

DEFINITION 3. *A lower bound of E exceeded by no lower bound of E is called the greatest lower bound or infimum of E, denoted by*

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6. By $\{a, b, c, \ldots\}$ is meant the set with elements $a, b, c, \ldots$

inf E. *An upper bound of E exceeding no upper bound of E is called the least upper bound or supremum of E, denoted by* sup E.

Remark 1. Put somewhat differently, if $\lambda = \inf E$ and λ' is any lower bound of E, then $\lambda' \leqslant \lambda$, while if $\Lambda = \sup E$ and Λ' is any upper bound of E, then $\Lambda \leqslant \Lambda'$. Equivalently, if $\lambda = \inf E$, then $\lambda \leqslant x$ for all $x \in E$, and given any $\varepsilon > 0$, there is an $x_0 = x_0(\varepsilon) \in E$ such that $x_0 < \lambda + \varepsilon$ (otherwise we could find a lower bound of E greater than λ). Similarly, if $\Lambda = \sup E$, then $x \leqslant \Lambda$ for all $x \in E$, and given $\varepsilon > 0$, there is an $x_0 \in E$ such that $x_0 > \Lambda - \varepsilon$.

Remark 2. According to Theorem 1.3, if min E exists, so does inf E and $\inf E = \min E$, while if max E exists, so does sup E and $\sup E = \max E$.

PROBLEMS

1. By a rational number is meant a number of the form p/q, where p and $q \neq 0$ are integers. Prove that $\sqrt{2}$ is irrational.

2. What is the rational number represented by each of the following decimals (the underlined block of digits repeats itself indefinitely):

a) $0.4\underline{17}$; b) $2.\underline{331}$; c) $0.91\underline{9}$?

3. Find max E, min E, sup E and inf E if

$$E = \{a, a^2, a^3, \ldots\}$$

and $0 \leqslant a \leqslant 1$. What happens if $a > 1$? Discuss the case of negative a.

CHAPTER 2

Limits

6. Basic Concepts

We now introduce one of the most important notions of mathematical analysis, namely that of a limit:

DEFINITION 1. *Let $f(x)$ be a function defined in some neighborhood of a point x_0, but not necessarily at x_0 itself. Then $f(x)$ is said to approach the limit c as x approaches x_0 (or to have the limit c at x_0) if, given any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that*

$$|f(x) - c| < \varepsilon$$

if $0 < |x - x_0| < \delta$.[1] *This fact is expressed by writing $f(x) \to c$ as $x \to x_0$ or*

$$\lim_{x \to x_0} f(x) = c.$$

Remark 1. In saying that $f(x)$ approaches a limit as $x \to x_0$, we mean that there is some c such that $f(x) \to c$ as $x \to x_0$.

Remark 2. We write $\delta(\varepsilon)$ to emphasize the dependence of the number δ on ε.

Remark 3. The limit of $f(x)$ at the point x_0 does not depend on the value of $f(x)$ at x_0, and in fact $f(x)$ may not even be defined at x_0.

Remark 4. Obviously, if $0 < |x - x_0| < \delta$ implies $|f(x) - c| < \varepsilon$, then so does $0 < |x - x_0| < \delta'$ for every $\delta' \leqslant \delta$. In particular, the limit of $f(x)$ at x_0 remains the same if we change

1. Equivalently, if $|x - x_0| < \delta$, $x \neq x_0$.

the value of $f(x)$ at any point $x_1 \neq x_0$. To see this, we need only choose $\delta' = \min\{\delta, |x_1 - x_0|\}$. In other words, the limit of $f(x)$ at x_0 depends only on the values of $f(x)$ in the "immediate vicinity" of x_0, not including the point x_0 itself.

THEOREM 2.1. *If $f(x)$ approaches a limit as $x \to x_0$, then this limit is unique.*

Proof. Suppose, to the contrary that

$$\lim_{x \to x_0} f(x) = c, \quad \lim_{x \to x_0} f(x) = c',$$

where $c \neq c'$. Choosing $\varepsilon = \frac{1}{2}|c - c'|$, we have

$$|f(x) - c| < \varepsilon \text{ if } 0 < |x - x_0| < \delta_1,$$

$$|f(x) - c'| < \varepsilon \text{ if } 0 < |x - x_0| < \delta_2$$

for suitable δ_1 and δ_2. Then

$$\begin{aligned} |c - c'| &= |c - f(x) + f(x) - c'| \\ &\leqslant |f(x) - c| + |f(x) - c'| < 2\varepsilon = |c - c'| \end{aligned}$$

if

$$0 < |x - x_0| < \min\{\delta_1, \delta_2\}.$$

But $|c - c'| < |c - c'|$ is impossible, and hence the theorem is proved.

Example 1. The function $f(x) = x$ approaches the limit x_0 as $x \to x_0$, as one would certainly hope! In fact, given any $\varepsilon > 0$ and choosing $\delta = \varepsilon$, we have $|x - x_0| < \varepsilon$ if $0 < |x - x_0| < \delta$.

Example 2. The function $f(x) = 5x$ approaches the limit 10 as $x \to 2$. In fact, given any $\varepsilon > 0$ and choosing $\delta = \varepsilon/5$, we have

$$|f(x) - 10| = |5x - 10| = 5|x - 2| < 5\delta = \varepsilon$$

if $0 < |x - 2| < \delta$.

Example 3. The function $f(x) = x^2 + 3x$ approaches the limit 4 as $x \to 1$. To see this, first assume that $0 \leqslant x \leqslant 2$, say, since only values of $f(x)$ in the "immediate vicinity" of the point $x = 1$ affect the limit at $x = 1$. Given any $\varepsilon > 0$, choose $\delta = 1$

if $\varepsilon > 6$ ($\delta > 1$ is not allowed, since $0 \leqslant x \leqslant 2$, i.e., $|x - 1| \leqslant 1$) and $\delta = \varepsilon/6$, so that $\delta \leqslant \varepsilon/6$ in any event. It follows that

$$\begin{aligned}|f(x) - 4| &= |x^2 + 3x - 4| = |(x + 4)(x - 1)| \\ &= |x + 4|\,|x - 1| \leqslant 6\,|x - 1| < 6\delta \leqslant \varepsilon\end{aligned}$$

if $0 < |x - 1| < \delta$.

Example 4. The function

$$f(x) = \frac{5x^2 - 10x}{x - 2},$$

which does not exist at $x = 2$, approaches the limit 10 as $x \to 2$. To see this, we note that

$$f(x) = \frac{5x(x - 2)}{x - 2} = 5x \quad \text{if } x \neq 2.$$

Therefore, by Example 1,

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} 5x = 10.$$

Remark. The fact that

$$\lim_{x \to x_0} f(x) = f(x_0)$$

in Examples 1–3 is no accident, as we shall see in the next chapter, since in both cases $f(x)$ is "continuous at x_0."

We now consider a less trivial limit:

THEOREM 2.2. *The function*

$$\frac{\sin x}{x}$$

approaches 1 *as* $x \to 0$.

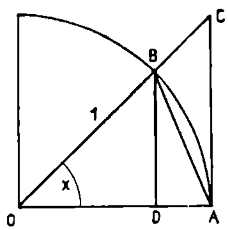

Fig. 8

Proof. First let $x > 0$, and consider the part of the circle of radius 1 with center at the origin which lies in the first quadrant. Then, as shown in Figure 8, the area S of the sector OAB lies between the areas of the triangles OAB and OAC. But the triangle OAB has base 1 and altitude $BD = \sin x$, while the triangle

OAC has base 1 and altitude $AC = \tan x$. Therefore

$$\tfrac{1}{2} \sin x < S < \tfrac{1}{2} \tan x.$$

Moreover, the area of the sector is just half its central angle times the square of the radius, so that

$$S = \tfrac{1}{2}x,$$

and hence

$$\sin x < x < \tan x. \tag{2.1}$$

Dividing (2.1) by $\sin x$ ($\sin x > 0$ since x is positive and sufficiently near 0, i.e., $0 < x < \pi/2$), we obtain

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}.$$

It follows that

$$1 > \frac{\sin x}{x} > \cos x$$

or

$$0 < 1 - \frac{\sin x}{x} < 1 - \cos x = 2 \sin^2 \frac{x}{2}$$

$$< 2 \sin \frac{x}{2} < \frac{2x}{2} = x.$$

Therefore

$$\left|1 - \frac{\sin x}{x}\right| < |x|$$

if $0 < x < \pi/2$. But both sides of this inequality are even, and hence the inequality continues to hold in the interval $0 > x > -\pi/2$. In other words,

$$\left|1 - \frac{\sin x}{x}\right| < |x| \text{ if } 0 < |x| < \frac{\pi}{2}.$$

Given any $\varepsilon > 0$, choose $\delta = \pi/2$ if $\varepsilon > \pi/2$ and $\delta = \varepsilon$ if $\varepsilon \leqslant \pi/2$. Then

$$\left|1 - \frac{\sin x}{x}\right| < \varepsilon$$

if $|x - 0| = |x| < \delta$, $x \neq 0$, and the theorem is proved.

The next example shows that a function need not have a limit at any point whatsoever!

Example 5. The Dirichlet function $D(x)$ introduced on p. 8 does not have a limit at any point. In fact, suppose to the contrary that $D(x)$ has a limit c at the point x_0. Then, given $\varepsilon = \frac{1}{2}$, we can find a number $\delta > 0$ such that $0 < |x - x_0| < \delta$ implies

$$|D(x) - c| < \tfrac{1}{2}. \tag{2.2}$$

But the interval $|x - x_0| < \delta$ contains a rational point $x_1 \neq x_0$ and an irrational point $x_2 \neq x_0$ (in fact, infinitely many of each). It follows from (2.2) that

$$|D(x_1) - c| = |1 - c| < \tfrac{1}{2},$$

$$|D(x_2) - c| = |0 - c| < \tfrac{1}{2},$$

and hence

$$1 = |1 - c + c| \leqslant |1 - c| + |c| < \tfrac{1}{2} + \tfrac{1}{2} = 1.$$

Since this is impossible, $D(x)$ has no limit at x_0, as asserted.

THEOREM 2.3. *If $f(x)$ has a positive (negative) limit at x_0, then there is a neighborhood of x_0 such that $f(x)$ is positive (negative) at every point of the neighborhood, except possibly x_0 itself.*

Proof. Let c be the limit of $f(x)$ at x_0, and suppose $c > 0$. Setting $\varepsilon = c/2$, we choose $\delta > 0$ such that $0 < |x - x_0| < \delta$ implies

$$|f(x) - c| < \frac{c}{2}.$$

But then

$$-\frac{c}{2} < f(x) - c < \frac{c}{2}$$

or

$$\frac{c}{2} < f(x) < \frac{3c}{2}, \tag{2.3}$$

i.e., $f(x) > 0$. If $c < 0$, choose $\varepsilon = -c/2$. Then (2.3) is replaced by

$$\frac{3c}{2} < f(x) < \frac{c}{2},$$

and $f(x) > 0$.

Remark. In other words, the values of a function $f(x)$ in a neighborhood of x_0 have the same sign as the limit of $f(x)$ at x_0, provided the latter exists and is nonzero.

DEFINITION 2. *A function $f(x)$ with domain X is said to be bounded on (or in) the set $E \subset X$ if there is a positive number M such that*

$$|f(x)| \leqslant M$$

for all $x \in E$. Otherwise, $f(x)$ is said to be unbounded on E.

Remark 1. Obviously, $f(x)$ is bounded on E if and only if the set of numbers $y = f(x)$, $x \in E$ is bounded.

Remark 2. If $f(x)$ is unbounded on E, then, given any $M > 0$, no matter how large, there is a point $x' \in E$ such that

$$|f(x')| > M.$$

Remark 3. If $f(x)$ is bounded on E, then

$$-M \leqslant f(x) \leqslant M$$

for some $M > 0$ and all $x \in E$. Geometrically, this means that the graph of $f(x)$ lies between the lines $y = -M$ and $y = M$ parallel to the x-axis. Figure 9 shows the graph of a function bounded

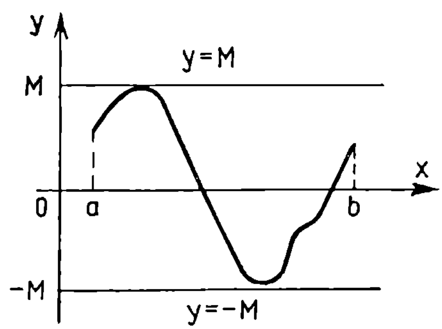

Fig. 9

in the interval $[a, b]$. Obviously, a function is also bounded if its graph lies between *any* two lines parallel to the x-axis, i.e., if

$$m \leqslant f(x) \leqslant M.$$

Example 6. The trigonometric functions $\sin x$ and $\cos x$ are bounded on the whole real axis, since

$$-1 \leqslant \sin x \leqslant 1, \quad -1 \leqslant \cos x \leqslant 1$$

for all x.

Example 7. The function

$$f(x) = \frac{x^2}{1 + x^2}$$

is bounded on the whole real axis, since $0 \leqslant f(x) \leqslant 1$ for all x.

Example 8. The function x^2 is bounded by the number $M = 25$ in the interval $[0, 5]$, but is unbounded on the whole real axis.

DEFINITION 3. *By a deleted neighborhood of x_0 is meant a neighborhood of x_0 minus the point x_0 itself.*

THEOREM 2.4. *If $f(x)$ approaches a limit as $x \to x_0$, then $f(x)$ is bounded in some deleted neighborhood of x_0.*

Proof. Let c be the limit of $f(x)$ at x_0. Choosing $\varepsilon = 1$, we find a number $\delta > 0$ such that $0 < |x - x_0| < \delta$ implies

$$|f(x) - c| < 1,$$

and hence

$$|f(x)| - |c| \leqslant |f(x) - c| < 1,$$

$$|f(x)| < |c| + 1 = M,$$

as required.

Example 9. The function

$$f(x) = \frac{1}{x} \quad (x \neq 0)$$

is unbounded in any deleted neighborhood of the point $x = 0$, and hence cannot have a limit as $x = 0$.

Example 10. The Dirichlet function $D(x)$ is bounded in a neighborhood of any point x_0, but has no limit at x_0. Hence the converse of Theorem 2.4 is false.

We conclude this section by giving the geometrical interpretation of a limit. Suppose that $|f(x) - c| < \varepsilon$ if

$$0 < |x - x_0| < \delta.$$

Then

$$-\varepsilon < f(x) - c < \varepsilon$$

or

$$c - \varepsilon < f(x) < c + \varepsilon,$$

and the graph of $f(x)$ lies in the strip between the lines $y = c - \varepsilon$ and $y = c + \varepsilon$ parallel to the x-axis. In fact, by requiring that the graph of $f(x)$ lie in this strip, we can determine an appropriate value of δ. The situation is illustrated by Figure 10.

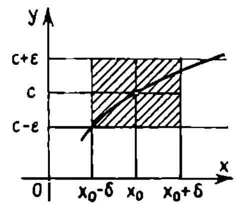

Fig. 10

PROBLEMS

1. If

$$f(x) = \frac{x^2 - 1}{x^2 + 1},$$

prove that

$$\lim_{x \to 2} f(x) = \tfrac{3}{5}.$$

Find a value of δ such that $0 < |x - 2| < \delta$ implies $|f(x) - \frac{3}{5}| < 0.001$.

2. Prove that $\sin x$ has the same sign as x in some deleted neighborhood of $x = 0$.

3. Which of the following limits exist:

$$\text{a) } \lim_{x \to 0} \frac{x}{x}; \quad \text{b) } \lim_{x \to 0} \frac{\sin \dfrac{1}{x}}{\sin \dfrac{1}{x}}; \quad \text{c) } \lim_{x \to 0} \frac{|x|}{x}?$$

4. Which of the following functions are bounded on the whole real line:

a) $y = \dfrac{1 - x^2}{1 + x^2}$; b) $y = \dfrac{1 + x^2}{x^2}$; c) $y = \dfrac{1}{x} \cos \dfrac{1}{x}$;

d) $y = [x]$; e) $y = \{x\}$.

(The symbols $[x]$ and $\{x\}$ have the same meaning as in Example 5, p. 18).

5. Prove that

$$y = \sin \frac{1}{x}$$

is bounded in a deleted neighborhood of $x = 0$ but has no limit at $x = 0$ (cf. Example 10, p 31.).

6. Give an example of a function which is periodic and unbounded.

7. Algebraic Properties of Limits

LEMMA 1. *The function* $f(x) \to a$ *as* $x \to x_0$ *if and only if* $f(x) = a + \alpha(x)$, *where* $\alpha(x) \to 0$ *as* $x \to x_0$.

Proof. If $f(x) \to a$ as $x \to x_0$, then, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - a| = |\alpha(x)| = |\alpha(x) - 0| < \varepsilon$$

if $0 < |x - x_0| < \delta$, and hence $\alpha(x) \to 0$ as $x \to x_0$. The converse follows by virtually the same argument.

LEMMA 2. *If* $\alpha(x) \to 0$ *and* $\beta(x) \to 0$ *as* $x \to x_0$, *then* $\alpha(x) \pm \beta(x) \to 0$ *as* $x \to x_0$.

Proof. Given any $\varepsilon > 0$, choose δ_1 and δ_2 such that

$$|\alpha(x)| < \frac{\varepsilon}{2} \text{ if } 0 < |x - x_0| < \delta_1,$$

$$|\beta(x)| < \frac{\varepsilon}{2} \text{ if } 0 < |x - x_0| < \delta_2.$$

Then

$$|\alpha(x) \pm \beta(x)| \leqslant |\alpha(x)| + |\beta(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

if $0 < |x - x_0| < \min\{\delta_1, \delta_2\}$.

COROLLARY. *If* $\alpha_1(x) \to 0, \ldots, \alpha_n(x) \to 0$ *as* $x \to x_0$, *then* $\alpha_1(x) \pm \cdots \pm \alpha_n(x) \to 0$ *as* $x \to x_0$.

Proof. Apply Lemma 2 repeatedly.

LEMMA 3. *If* $f(x)$ *is bounded in a deleted neighborhood of* x_0 *and if* $\alpha(x) \to 0$ *as* $x \to x_0$, *then* $f(x)\alpha(x) \to 0$ *as* $x \to x_0$.

Proof. By hypothesis, there are numbers $M > 0$ and $\delta_1 > 0$ such that $|f(x)| \leqslant M$ if $0 < |x - x_0| < \delta_1$. Moreover, given any $\varepsilon > 0$, there is a $\delta_2 > 0$ such that

$$|\alpha(x)| < \frac{\varepsilon}{M}$$

if $0 < |x - x_0| < \delta_2$. Therefore

$$|f(x)\alpha(x)| = |f(x)|\,|\alpha(x)| < M\frac{\varepsilon}{M} = \varepsilon$$

if $0 < |x - x_0| < \min\{\delta_1, \delta_2\}$.

COROLLARY. *If* $f(x) \to a$ *and* $\alpha(x) \to 0$ *as* $x \to x_0$, *then* $f(x)\alpha(x) \to 0$ *as* $x \to x_0$.

Proof. Use Theorem 2.4.

THEOREM 2.5. *If* $f(x) \to a$ *and* $g(x) \to b$ *as* $x \to x_0$, *then* $f(x) \pm g(x) \to a \pm b$ *as* $x \to x_0$.

Proof. According to Lemma 1, $f(x) = a + \alpha(x)$, $g(x) = b + \beta(x)$, where $\alpha(x) \to 0$ and $\beta(x) \to 0$ as $x \to x_0$. Therefore

$$f(x) \pm g(x) = a \pm b + [\alpha(x) \pm \alpha(x)],$$

where the expression in brackets approaches 0 as $x \to x_0$, by Lemma 2. Hence, by Lemma 1 again, $f(x) \pm g(x) \to a \pm b$ as $x \to x_0$.

COROLLARY. *If* $f_1(x) \to a_1, \ldots, f_n(x) \to a_n$ *as* $x \to x_0$, *then*

$$f_1(x) \pm \cdots \pm f_n(x) \to a_1 \pm \cdots \pm a_n$$

as $x \to x_0$.

Proof. Use the corollary to Lemma 2, or more simply, apply Theorem 2.5 repeatedly.

THEOREM 2.6. *If* $f(x) \to a$ *and* $g(x) \to b$ *as* $x \to x_0$, *then* $f(x)\,g(x) \to ab$ *as* $x \to x_0$.

Proof. According to Lemma 1, $f(x) = a + \alpha(x)$, $g(x) = b + \beta(x)$, where $\alpha(x) \to 0$ and $\beta(x) \to 0$ as $x \to x_0$. Therefore

$$f(x)\,g(x) = ab + [b\alpha(x) + a\beta(x) + \alpha(x)\,\beta(x)],$$

where according to the corollary to Lemma 2 and Lemma 3 and its corollary, the expression in brackets approaches 0 as $x \to x_0$.

COROLLARY 1. *If* $f(x) \to a_1, \ldots, f_n(x) \to a_n$ *as* $x \to x_0$, *then the product* $f_1(x) \cdots f_n(x)$ *approaches* $a_1 \cdots a_n$ *as* $x \to x_0$. *In particular, if* $f(x) \to a$ *as* $x \to x_0$, *then* $[f(x)]^n \to a^n$ *as* $x \to x_0$.

Proof. Apply Theorem 2.6 repeatedly.

COROLLARY 2. *If* $f(x) \to a$ *as* $x \to x_0$, *then* $cf(x) \to ca$ *as* $x \to x_0$.

Proof. Obviously the function $f(x) \equiv c$ approaches c as $x \to x_0$.[2]

THEOREM 2.7. *If* $f(x) \to a$ *and* $g(x) \to b \neq 0$ *as* $x \to x_0$, *then*

$$\frac{f(x)}{g(x)} \to \frac{a}{b}$$

as $x \to x_0$.

Proof. Writing

$$\frac{f(x)}{g(x)} - \frac{a}{b} = \frac{1}{bg(x)}\,[bf(x) - ag(x)],$$

we examine each factor on the right separately. According to Theorem 2.6, Corollary 2, $bg(x) \to b^2$ as $x \to x_0$, where $b^2 > 0$ since $b \neq 0$. Therefore, by the inequality (2.3), p. 28,

$$\tfrac{1}{2}b^2 < bg(x) < \tfrac{3}{2}b^2$$

---

2. The symbol $\equiv$ means "(is) identically equal to."

in a sufficiently small deleted neighborhood of x_0. But then

$$0 < \frac{1}{bg(x)} < \frac{1}{\frac{1}{2}b^2} = \frac{2}{b^2}.$$

and hence $1/bg(x)$ is bounded in a deleted neighborhood of x_0. Moreover, by Theorems 2.5 and 2.6,

$$bf(x) - ag(x) \to ba - ab = 0.$$

Therefore, by Lemma 3,

$$\frac{f(x)}{g(x)} - \frac{a}{b} \to 0 \text{ as } x \to x_0.$$

The theorem is now a consequence of Lemma 1.

In the following examples, we draw freely on all the results of this section.

Example 1. Find the limit of $x^2 - 2x$ *as* $x \to 4$.
Solution. Clearly

$$\lim_{x\to 4} (x^2 - 2x) = \lim_{x\to 4} x^2 - \lim_{x\to 4} 2x = 4^2 - 2 \cdot 4 = 8.$$

Example 2. Find the limit as $x \to 0$ *of*

$$\frac{\sin 5x}{2x}.$$

Solution. Since

$$\frac{\sin 5x}{2x} = \frac{5}{2}\frac{\sin 5x}{5x} = \frac{5}{2}\frac{\sin t}{t},$$

where $t = 5x \to 0$ as $x \to 0$, it follows from Theorem 2.2 that

$$\lim_{x\to 0}\frac{\sin 5x}{2x} = \frac{5}{2}\lim_{t\to 0}\frac{\sin t}{t} = \frac{5}{2}.$$

Example 3. Find the limit as $x \to 0$ *of*

$$\frac{1 - \cos x}{x^2}.$$

Solution. Clearly,

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{2 \sin^2 \frac{x}{2}}{x^2} = \lim_{x \to 0} \frac{1}{2} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 = \frac{1}{2}.$$

Example 4. Find the limit as $x \to 2$ of

$$\frac{x^3 - 8}{x^2 - 4}.$$

Solution. Since only values at points $x \neq 2$ matter, we have

$$\frac{x^3 - 8}{x^2 - 4} = \frac{(x - 2)(x^2 + 2x + 4)}{(x - 2)(x + 2)} = \frac{x^2 + 2x + 4}{x + 2},$$

and hence

$$\lim_{x \to 0} \frac{x^3 - 8}{x^2 - 4} = \lim_{x \to 2} \frac{x^2 + 2x + 4}{x + 2}$$

$$= \frac{\lim\limits_{x \to 2} (x^2 + 2x + 4)}{\lim\limits_{x \to 2} (x + 2)} \doteq \frac{2^2 + 2 \cdot 2 + 4}{2 + 2} = 3.$$

PROBLEMS

1. Find the following limits:

a) $\lim\limits_{x \to 4} \left[2(x + 3) - \frac{x}{x - 2} \right]$; b) $\lim\limits_{x \to 1} \left(x^5 - 5x + 2 + \frac{1}{x} \right)$;

c) $\lim\limits_{x \to 0} \frac{3x^3 + 2x^2 - x}{5x}$; d) $\lim\limits_{t \to 1} \frac{t(t - 1)}{2(t^2 - 1)}$;

e) $\lim\limits_{x \to 5} \frac{x^3 - 25}{x - 5}$; f) $\lim\limits_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - 12x + 20}$;

g) $\lim_{x \to -2} \dfrac{2x^2 + 5x - 7}{3x^2 - x - 2}$; h) $\lim_{x \to 1} \dfrac{2x^3 - 2x^2 + x - 1}{x^3 - x^2 + 3x - 3}$;

i) $\lim_{x \to -1} \dfrac{2x^3 + 2x^2 + 3x + 3}{x^3 + x^2 + x + 1}$ j) $\lim_{h \to 0} \dfrac{(x + h)^3 - x^3}{h}$.

2. Find the following limits:

a) $\lim_{x \to 0} \dfrac{x}{\sin x}$; b) $\lim_{x \to 0} \dfrac{x}{\sin 3x}$; c) $\lim_{x \to 0} \dfrac{\sin (1 + x)}{1 + x}$;

d) $\lim_{x \to 0} \dfrac{\sin mx}{\sin nx}$; e) $\lim_{x \to 0} \dfrac{\sin^2 x}{x}$; f) $\lim_{x \to 0} \dfrac{\sin^3 \dfrac{x}{4}}{x^3}$;

g) $\lim_{x \to 0} \dfrac{1 - \cos x}{5x}$.

3. Prove that

$$\lim_{x \to 0} \cos x = 1, \quad \lim_{x \to 0} \tan x = 0.$$

8. Limits Relative to a Set. One-Sided Limits

DEFINITION 1. *A (finite) point x_0 is said to be a limit point of a set E if every neighborhood of x_0 contains infinitely many points of E.*

Remark. Thus a set containing only a finite number of points cannot have a limit point.

We now generalize the definition of a limit given in Sec. 6:

DEFINITION 2. *Let $f(x)$ be a function with domain X, and let E be a subset of X with x_0 as a limit point (x_0 need not belong to either E or X). Then $f(x)$ is said to approach the limit c as x approaches x_0 relative to E (or to have the limit c at x_0 relative to E) if given any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that*

$$|f(x) - c| < \varepsilon$$

if $0 < |x - x_0| < \delta$, $x \in E$. *This fact is expressed by writing* $f(x) \to c$ *as* $x \to x_0$, $x \in E$ *or*

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x) = c.$$

Remark 1. If E is a deleted neighborhood of x_0, the qualifying conditions "relative to E" and "$x \in E$" are superfluous and can be dropped. Thus Definition 2 includes Definition 1, p. 24 as a special case.

Remark 2. It is clear that all the results of Sec.7 apply equally well to limits relative to a set E.

If E consists of all x such that $0 < x_0 - x < \delta$ (for some $\delta > 0$), we write

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x) = \lim_{x \to x_0 -} f(x) = f(x_0 -)$$

and call $f(x_0-)$ the *left-hand limit* of $f(x)$ at x_0. Similarly, if E consists of all x such that $0 < x - x_0 < \delta$, we write

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0 +} f(x) = f(x_0 +)$$

and call $f(x_0+)$ the *right-hand limit* of $f(x)$ at x_0. Clearly, if a function $f(x)$ is defined in an interval with end points a and b ($a < b$), we can only talk about a right-hand limit at a and a left-hand limit at b. The limits $f(x_0-)$ and $f(x_0+)$ are called *one-sided limits*, for an obvious reason.

DEFINITION 3. *By the union of two sets* E_1 *and* E_2, *denoted by* $E_1 \cup E_2$, *is meant the set of points belonging to at least one of the sets* E_1 *and* E_2. *If* E_1 *and* E_2 *have no points in common, they are said to be disjoint.*

THEOREM 2.8. *Suppose* $f(x)$ *is defined on two disjoint sets* E_1 *and* E_2, *each having* x_0 *as a limit point, and let* $E = E_1 \cup E_2$. *Then*

$$f(x) \to c \text{ as } x \to x_0, \quad x \in E \tag{2.4}$$

if and only if

$$\begin{aligned} f(x) \to c \text{ as } x \to x_0, \quad x \in E_1, \\ f(x) \to c \text{ as } x \to x_0, \quad x \in E_2. \end{aligned} \tag{2.5}$$

Proof. Put somewhat differently, the theorem asserts that

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x)$$

exists if and only if the limits

$$\lim_{\substack{x \to x_0 \\ x \in E_1}} f(x), \quad \lim_{\substack{x \to x_0 \\ x \in E_2}} f(x)$$

exist and are equal, in which case

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x) = \lim_{\substack{x \to x_0 \\ x \in E_1}} f(x) = \lim_{\substack{x \to x_0 \\ x \in E_2}} f(x).$$

If (2.4) holds, then, given any $\varepsilon > 0$, there is a $\delta > 0$ such that $0 < |x - x_0| < \delta$, $x \in E$ implies $|f(x) - c| < \varepsilon$. But then $0 < |x - x_0| < \delta$, $x \in E_1$ implies $|f(x) - c| < \varepsilon$ and so does $0 < |x - x_0| < \delta$, $x \in E_2$, since $E_1 \subset E$, $E_2 \subset E$.

Conversely, if (2.5) holds, then given any $\varepsilon > 0$, there are positive numbers δ_1 and δ_2 such that $0 < |x - x_1| < \delta_1$, $x \in E_1$ implies $|f(x) - c| < \varepsilon$ and so does $0 < |x - x_0| < \delta_2$, $x \in E_2$. But then $0 < |x - x_0| < \min\{\delta_1, \delta_2\}$ implies $|f(x) - c| < \varepsilon$, since $E = E_1 \cup E_2$, and the proof is complete.

COROLLARY 1. *The function $f(x) \to c$ as $x \to x_0$ if and only if*

$$\lim_{x \to x_0 -} f(x) = \lim_{x \to x_0 +} f(x) = f(x_0 -) = f(x_0 +) = c.$$

COROLLARY 2. *Let R be the set of rational numbers and I the set of irrational numbers. Then $f(x) \to c$ as $x \to x_0$ if and only if*

$$\lim_{\substack{x \to x_0 \\ x \in R}} f(x) = \lim_{\substack{x \to x_0 \\ x \in I}} f(x) = c.$$

Example 1. Find the limit as $x \to 2$ of the function

$$f(x) = \begin{cases} x^2 & \text{if } x < 2, \\ x^3 - x - 2 & \text{if } x > 2. \end{cases}$$

Solution. Clearly

$$\lim_{x\to 2-} f(x) = \lim_{x\to 2-} x^2 = 4,$$

$$\lim_{x\to 2+} f(x) = \lim_{x\to 2+} (x^3 - x - 2) = 4,$$

and hence, by Corollary 1,

$$\lim_{x\to 2} f(x) = 4.$$

Example 2. If $f(x) = x^2$, $x \in [0, 2]$, then

$$\lim_{x\to 0+} f(x) = 0, \quad \lim_{x\to 2-} f(x) = 4,$$

but

$$\lim_{x\to 0} f(x), \quad \lim_{x\to 0-} f(x), \quad \lim_{x\to 2} f(x), \quad \lim_{x\to 2+} f(x)$$

all fail to exist.

Example 3. Give another proof of the fact that the Dirichlet function $D(x)$ does not have a limit anywhere.[3]

Solution. If x_0 is arbitrary, then obviously

$$\lim_{\substack{x\to x_0\\ x\in R}} D(x) = 1, \quad \lim_{\substack{x\to x_0\\ x\in I}} D(x) = 0.$$

Therefore, according to Corollary 2,

$$\lim_{x\to x_0} D(x)$$

does not exist.

Example 4. Find every point at which the function

$$f(x) = \begin{cases} x^2 - x \textit{ if } x \textit{ is rational,} \\ 2x - 2 \textit{ if } x \textit{ is irrational} \end{cases}$$

has a limit.

Solution. We have

$$\lim_{\substack{x\to x_0\\ x\in R}} f(x) = \lim_{x\to x_0} (x^2 - x) = x_0^2 - x_0,$$

$$\lim_{\substack{x\to x_0\\ x\in I}} f(x) = \lim_{x\to x_0} (2x - 2) = 2x_0 - 2,$$

---

3. See Example 5, p. 28.

and hence, by Corollary 2,

$$\lim_{x \to x_0} f(x)$$

exists if and only if $x_0^2 - x_0 = 2x_0 - 2$, i.e., if and only if $x_0 = 1$ or $x_0 = 2$.

THEOREM 2.9. *If $f(x)$, $g(x)$ and $h(x)$ satisfy the inequalities*

$$f(x) \leqslant g(x) \leqslant h(x)$$

for all $x \in E$, and if

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x) = \lim_{\substack{x \to x_0 \\ x \in E}} h(x) = c, \tag{2.6}$$

then

$$\lim_{\substack{x \to x_0 \\ x \in E}} g(x) = c. \tag{2.7}$$

Proof. If (2.6) holds, then, given any $\varepsilon > 0$, there are positive numbers δ_1 and δ_2 such that $0 < |x - x_0| < \delta_1$, $x \in E$ implies $|f(x) - c| < \varepsilon$ and hence

$$-\varepsilon < f(x) - c < \varepsilon,$$

while $0 < |x - x_0| < \delta_2$, $x \in E$ implies $|h(x) - c| < \varepsilon$ and hence

$$-\varepsilon < h(x) - c < \varepsilon.$$

But then $0 < |x - x_0| < \min\{\delta_1, \delta_2\}$ implies

$$-\varepsilon < f(x) - c \leqslant g(x) - c \leqslant h(x) - c < \varepsilon$$

and hence $|g(x) - c| < \varepsilon$, which is equivalent to (2.7).

Remark. If

$$f(x) \leqslant g(x), \quad x \in E \tag{2.8}$$

and if $f(x)$ and $g(x)$ approach limits as $x \to x_0$, $x \in E$, then

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x) \leqslant \lim_{\substack{x \to x_0 \\ x \in E}} g(x). \tag{2.9}$$

In fact, $g(x) - f(x) \geqslant 0$, $x \in E$, and hence

$$\lim_{\substack{x \to x_0 \\ x \in E}} g(x) - \lim_{\substack{x \to x_0 \\ x \in E}} f(x) = \lim_{\substack{x \to x_0 \\ x \in E}} [g(x) - f(x)] \geqslant 0,$$

by an obvious modification of Theorem 2.3. If (2.8) is replaced by

$$f(x) < g(x), \quad x \in E,$$

we can again only deduce (2.9), *and not*

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x) < \lim_{\substack{x \to x_0 \\ x \in E}} g(x).$$

For example, if $f(x) = 0$, $g(x) = x^2$, then $f(x) < g(x)$ for all nonzero x, but

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = 0.$$

PROBLEMS

1. Find the left and right-hand limits of the function $y = [x]$ at the points $x = -2, -1, 0, 1, 2$ ($[x]$ is the integral part of x).

2. Find the left and right-hand limits of the function $y = \{x\}$ at the points $x = 0, 1, 2, 3$ ($\{x\}$ is the fractional part of x).

3. Find the left and right-hand limits of the functions

$$f(x) = \frac{x}{x}, \quad \varphi(x) = \frac{|x|}{x}$$

at the point $x = 0$.

4. Find the left and right-hand limits of the function

$$f(x) = \frac{x^3 - 1}{x - 1}$$

at the point $x = 1$.

5. Find the left and right-hand limits of the function

$$f(x) = \frac{t}{\sqrt{|\sin t|}}$$

at the point $t = 0$.

6. Find the left-hand limit of the function

$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } -\infty < x < 0, \\ \sin \dfrac{1}{x} & \text{if } \quad 0 < x < \infty \end{cases}$$

at the point $x = 0$. Does the right-hand limit exist?

9. Infinite Limits. Indeterminate Forms

We now permit infinity as a possible limiting value of $f(x)$:[4]

DEFINITION. *Let $f(x)$ be a function defined in some deleted neighborhood of the point x_0. Then $f(x)$ is said to approach infinity as $x \to x_0$ if, given any $M > 0$, there exists a $\delta = \delta(M) > 0$ such that*

$$|f(x)| > M$$

if $0 < |x - x_0| < \delta$. This fact is expressed by writing $f(x) \to \infty$ as $x \to x_0$ or

$$\lim_{x \to x_0} f(x) = \infty .$$

THEOREM 2.10. *The function $f(x) \to \infty$ as $x \to x_0$ if and only if $1/f(x) \to 0$ as $x \to x_0$.*

Proof. If $1/f(x) \to 0$ as $x \to x_0$, then, given any $M > 0$, there is a $\delta > 0$ such that

$$\left| \frac{1}{f(x)} \right| < \frac{1}{M}, \text{ i.e., } |f(x)| > M$$

if $0 < |x - x_0| < \delta$, and hence $f(x) \to \infty$ as $x \to x_0$. The converse follows by detailed reversal of steps.

THEOREM 2.11. *If $f(x)$ is bounded in a deleted neighborhood of x_0 and if $g(x) \to \infty$ as $x \to x_0$, then*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0, \quad \lim_{x \to x_0} \frac{g(x)}{f(x)} = \infty .$$

4. But not as a possible value of $f(x)$ itself.

Proof. Apply the preceding theorem and Lemma 3, p. 33.

COROLLARY 1. *If* $f(x) \to c \neq 0$ *as* $x \to x_0$ *(where the value* $c = \infty$ *is allowed), and if* $g(x) \to \infty$ *as* $x \to x_0$, *then* $f(x)g(x) \to \infty$ *as* $x \to x_0$.

Proof. The function $1/f(x)$ approaches a finite limit as $x \to x_0$ (zero if $c = \infty$), and hence is bounded in a deleted neighborhood of x_0, by Theorem 2.4.

COROLLARY 2. *If* $f(x) \to c \neq 0$ *as* $x \to x_0$ *(where the value* $c = \infty$ *is allowed), and if* $g(x) \to 0$ *as* $x \to x_0$, *then* $f(x)/g(x) \to \infty$ *as* $x \to x_0$.

COROLLARY 3. *If* $f(x) \to c \neq \infty$ *and* $g(x) \to \infty$ *as* $x \to x_0$, *then* $f(x)/g(x) \to 0$ *as* $x \to x_0$.

Thus we have shown, crudely speaking, that

$$c \cdot \infty = \infty \text{ if } c \neq 0,$$

$$\frac{c}{0} = \infty \text{ if } c \neq 0$$

and

$$\frac{c}{\infty} = 0 \text{ if } c \neq \infty,$$

but we still know nothing at all about

$$0 \cdot \infty, \quad \frac{0}{0}, \quad \frac{\infty}{\infty}. \tag{2.10}$$

In fact, as the following examples show, these expressions can take any finite value, approach infinity or even fail to exist. For this reason, they are called *indeterminate forms*.

Example 1. If $f(x) = cx$ and $g(x) = x$, then

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = c.$$

Example 2. If $f(x) = x$ and $g(x) = x^2$, then

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{1}{x} = \infty.$$

Example 3. If $f(x) = xD(x)$, where $D(x)$ is the Dirichlet function (see p. 8), and $g(x) = x$, then

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} D(x)$$

fails to exist (see Example 5, p. 28).

Example 4. It follows from Examples 1–3 that 0/0 can take any finite value, approach infinity or even fail to exist. The same is true of $0 \cdot \infty$ and ∞/∞. In the first case, we need only write

$$\frac{f(x)}{g(x)} = f(x) \frac{1}{g(x)}$$

and note that $1/g(x) \to \infty$ if $g(x) \to 0$, while in the second case we write

$$\frac{f(x)}{g(x)} = \frac{\dfrac{1}{g(x)}}{\dfrac{1}{f(x)}},$$

observing that $1/f(x) \to \infty$, $1/g(x) \to \infty$ if $f(x) \to 0$, $g(x) \to 0$.

Thus, to find the limits of indeterminate forms, we must resort to various tricks. This is how we found

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

in Theorem 2.2, and

$$\lim_{x\to 2} \frac{x^3 - 8}{x^2 - 4} = 3$$

in Example 4, p. 36 (both of the form 0/0).

THEOREM 2.12. *If $f(x) \to \infty$ and $g(x) \to c \neq \infty$ as $x \to x_0$, then $f(x) + g(x) \to \infty$ as $x \to x_0$. This conclusion holds true for $c = \infty$, provided $f(x)$ and $g(x)$ have the same sign in a deleted neighborhood of x_0.*

Proof. To prove the first assertion, we note that

$$f(x) + g(x) = f(x)\left[1 + \frac{g(x)}{f(x)}\right],$$

where $g(x)/f(x) \to 0$ and hence

$$1 + \frac{g(x)}{f(x)} \to 1.$$

The second assertion follows from the inequality

$$|f(x) + g(x)| > |f(x)|,$$

valid if $f(x)$ and $g(x)$ have the same sign.

Thus we have shown, roughly speaking, that

$$\infty + c = \infty \text{ if } c \neq \infty$$

and $\infty + \infty = \infty$, but we still know nothing at all about the case $\infty - \infty$. In fact, as the following examples show, this expression is an indeterminate form, just like (2.10), which can take any finite value, approach infinity or even fail to exist.

Example 5. If

$$f(x) = c + \frac{1}{x}, \quad g(x) = -\frac{1}{x},$$

then

$$\lim_{x \to 0} [f(x) + g(x)] = c.$$

Example 6. If

$$f(x) = \frac{2}{x}, \quad g(x) = -\frac{1}{x},$$

then

$$\lim_{x \to 0} [f(x) + g(x)] = \lim_{x \to 0} \frac{1}{x} = \infty.$$

Example 7. If

$$f(x) = D(x) + \frac{1}{x}, \quad g(x) = -\frac{1}{x},$$

then

$$\lim_{x \to 0} [f(x) + g(x)] = \lim_{x \to 0} D(x)$$

fails to exist.

The relation between functions approaching infinity and unbounded functions is revealed by

THEOREM 2.13. *If $f(x) \to \infty$ as $x \to x_0$, then $f(x)$ is unbounded in a deleted neighborhood of x_0, but the converse is false.*

Proof. If $f(x)$ is bounded in a deleted neighborhood of x_0, there are positive numbers M and δ such that $|f(x)| \leqslant M$ if $0 < |x - x_0| < \delta$. But then for this M there is no δ such that $|f(x)| > M$ if $0 < |x - x_0| < \delta$, i.e., $f(x) \to \infty$ as $x \to x_0$ is impossible.

The function

$$f(x) = \frac{D(x)}{x},$$

where $D(x)$ is the Dirichlet function, shows that the converse is false. In fact, $f(x)$ is unbounded in every deleted neighborhood of the origin (where it takes arbitrarily large values at points), but does not approach ∞ as $x \to 0$, since every such neighborhood contains (irrational) points where $f(x) = 0$.

Remark. To define a function approaching infinity as x approaches x_0 *relative to* E (where x_0 is a limit point of the set E), we need only generalize the definition on p. 43 in the same way as Definition 2, p. 37 generalizes Definition 1, p. 24. Thus

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x) = \infty$$

means that given any $M > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $|f(x)| > M$ if $0 < |x - x_0| < \delta$, $x \in E$.

PROBLEMS

1. Does the function

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \text{ is rational,} \\[2ex] -\dfrac{1}{x} & \text{if } x \text{ is irrational} \end{cases}$$

approach infinity as $x \to 0$?

2. Does the function

$$f(x) = \frac{1}{x} \cos \frac{1}{x}$$

approach infinity as $x \to 0$?

3. Find the following limits:

a) $\lim\limits_{x \to 2} \dfrac{x-2}{x^2 - 3x + 2}$; b) $\lim\limits_{x \to \pi} \dfrac{\tan x}{\sin 2x}$;

c) $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{\sin x - \cos x}{\cos 2x}$; d) $\lim\limits_{x \to 1} \dfrac{\sqrt[3]{x} - 1}{\sqrt{x} - 1}$;

e) $\lim\limits_{x \to 0} \dfrac{\sqrt[3]{1 + mx} - 1}{x}$.

4. Find the following limits:

a) $\lim\limits_{x \to 1} \left(\dfrac{1}{x-1}\right) - \left(\dfrac{2}{x^2 - 1}\right)$; b) $\lim\limits_{x \to 2} \left(\dfrac{1}{x-2} - \dfrac{12}{x^3 - 8}\right)$;

c) $\lim\limits_{x \to 0} \left(\dfrac{1}{\sin^2 x} - \dfrac{1}{4 \sin^2 \dfrac{x}{2}}\right)$; d) $\lim\limits_{x \to 1} (1 - x) \tan \dfrac{\pi x}{2}$;

e) $\lim\limits_{x \to \frac{\pi}{2}} \left(x - \dfrac{\pi}{2}\right) \tan x$.

10. Limits at Infinity

Next we allow the argument of $f(x)$, as well as $f(x)$ itself, to approach infinity (∞):

DEFINITION 1. *Given any $N > 0$, a set of the form $|x| > N$ is called a neighborhood of ∞.*[5]

DEFINITION 2. *The point ∞ is said to be a limit point of a set E if every neighborhood of ∞ contains infinitely many points of E.*

DEFINITION 3. *Let $f(x)$ be a function with domain X, and let E be a subset of X with ∞ as a limit point. Then $f(x)$ is said to approach the limit c as x approaches ∞ relative to E (or to have the limit c at ∞ relative to E) if, given any $\varepsilon > 0$, there exists a number $N = N(\varepsilon) > 0$ such that*

$$|f(x) - c| < \varepsilon$$

if $|x| > N$, $x \in E$. This fact is expressed by writing $f(x) \to c$ as $x \to \infty$, $x \in E$ or

$$\lim_{\substack{x \to \infty \\ x \in E}} f(x) = c.$$

Remark 1. If E is a neighborhood of ∞, the qualifying conditions "relative to E" and "$x \in E$" are superfluous and can be dropped.

Remark 2. To allow the case $c = \infty$, we change the phrase "given any $\varepsilon > 0$" and the inequality $|f(x) - c| < \varepsilon$ to "given any $M > 0$" and the inequality $|f(x)| > M$, just as in going from Definition 1, p. 24 to the definition on p. 43.

Remark 3. Obviously, if $|x| > N$ implies $|f(x) - c| < \varepsilon$, then so does $|x| > N'$ for every $N' \geqslant N$. In particular, the limit of $f(x)$ at ∞ remains the same if we change the values of $f(x)$ at any (finite) point x_0. To see this, we need only choose

$$N' = \max\{N, |x_0|\}.$$

---

5. The "point" ∞ is not regarded as a real number, and hence there is no need to talk about deleted neighborhoods. It is important not to confuse the symbols ∞ and $+\infty$ (see below).

In other words, the limit of $f(x)$ at ∞ depends only on the values of $f(x)$ in the "immediate vicinity" of ∞.

Remark 4. If E consists of all x such that $x < -N$ (for some $N > 0$), we write

$$\lim_{\substack{x \to \infty \\ x \in E}} f(x) = \lim_{x \to -\infty} f(x).$$

Similarly, if E consists of all x such that $x > N$, we write

$$\lim_{\substack{x \to \infty \\ x \in E}} f(x) = \lim_{x \to +\infty} f(x).$$

THEOREM 2.14. *The function $f(x) \to c$ as $x \to \infty$ if and only if $f^*(\xi) = f(1/\xi) \to c$ as $\xi \to 0$.*

Proof. If $c \neq \infty$, then, given any $\varepsilon > 0$, there is an $N > 0$ such that

$$|f(x) - c| < \varepsilon$$

if $|x| > N$. Therefore

$$\left| f\left(\frac{1}{1/x}\right) - c \right| = \left| f^*\left(\frac{1}{x}\right) - c \right| < \varepsilon$$

if $|x| > N$, i.e.,

$$|f^*(\xi) - c| < \varepsilon$$

if $|\xi| < \delta = 1/N$, and hence $f^*(\xi) \to c$ as $\xi \to \infty$. If $c = \infty$, then given any $M > 0$, there is an $N > 0$ such that

$$|f(x)| > M$$

if $|x| > N$. Therefore

$$\left| f\left(\frac{1}{1/x}\right) \right| = \left| f^*\left(\frac{1}{x}\right) \right| > M$$

if $|x| > N$, i.e.,

$$|f^*(\xi)| > M$$

if $|\xi| < \delta = 1/N$, and hence again $f^*(\xi) \to c$ as $\xi \to 0$. The converse follows by detailed reversal of steps.

COROLLARY. *The function* $f(x) \to c$ *as* $x \to \pm\infty$ *if and only if* $f^*(\xi) = f(1/\xi) \to c$ *as* $\xi \to 0\pm$.

Example 1. According to Theorem 2.14,

$$\lim_{x\to\infty} \frac{2x^2 + 7}{5x^2 + 3x - 4} = \lim_{x\to\infty} \frac{2 + \dfrac{7}{x^2}}{5 + \dfrac{3}{x} - \dfrac{4}{x^2}}$$

$$= \lim_{\xi\to 0} \frac{2 + 7\xi^2}{5 + 3\xi - 4\xi^2} = \frac{2}{5}.$$

Example 2. If

$$f(x) = \begin{cases} x & \text{if } x \geqslant 0, \\ \dfrac{1}{x} & \text{if } x < 0, \end{cases} \tag{2.11}$$

then according to the corollary,

$$\lim_{x\to-\infty} f(x) = \lim_{\xi\to 0-} \xi = 0, \quad \lim_{x\to+\infty} f(x) = \lim_{\xi\to 0+} \frac{1}{\xi} = \infty.$$

Remark 1. Using Theorem 2.14, we can deduce analogues of most of the results proved earlier. For example, the analogue of Theorem 2.6 states that if $f(x) \to a \neq \infty$ and $g(x) \to b \neq \infty$ as $x \to \infty$, then $f(x)\,g(x) \to ab$ as $x \to \infty$. This follows at once by noting that

$$\lim_{x\to\infty} f(x)\,g(x) = \lim_{\xi\to 0} f\left(\frac{1}{\xi}\right) g\left(\frac{1}{\xi}\right) = \lim_{\xi\to 0} f\left(\frac{1}{\xi}\right) \lim_{\xi\to 0} g\left(\frac{1}{\xi}\right)$$

$$= \lim_{x\to\infty} f(x) \lim_{x\to\infty} g(x).$$

Similarly, the analogue of Theorem 2.8, Corollary 1 states that $f(x) \to c$ as $x \to \infty$ if and only if

$$\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = c.$$

Remark 2. A further refinement is possible. We say that $f(x) \to -\infty$ as $x \to x_0$ (the values $x_0 = \infty$ and $\pm\infty$ are allowed) if $f(x) \to \infty$ as $x \to x_0$ and if $f(x) < 0$ for all x sufficiently near x_0. Similarly, $f(x) \to +\infty$ as $x \to x_0$ if $f(x) \to \infty$ as $x \to x_0$ and $f(x) > 0$ for all x sufficiently near x_0. By "sufficiently near x_0" we mean for all x in some deleted neighborhood of x_0 if x_0 is finite, in some neighborhood of ∞ if $x_0 = \infty$, for all $x < -N$ (for some $N > 0$) if $x_0 = -\infty$, and for all $x > N$ if $x_0 = +\infty$.

Example 3. If $f(x)$ is the function (2.11), then

$$\lim_{x \to 0-} f(x) = -\infty, \quad \lim_{x \to 0+} f(x) = 0.$$

Example 4. As $x \to \infty$, the function $1/x$ approaches ∞, but not $-\infty$ or $+\infty$.

PROBLEMS

1. Let

$$f(x) = \frac{a_0 x^m + a_1 x^{m-1} + \cdots + a_m}{b_0 x^n + b_1 x^{n-1} + \cdots + b_n},$$

where $a_0 \neq 0$, $b_0 \neq 0$. Prove that

$$\lim_{x \to \infty} f(x) = \begin{cases} 0 \text{ if } m < n. \\ \dfrac{a_0}{b_0} \text{ if } m = n, \\ \infty \text{ if } m > n, \end{cases}$$

2. Find the following limits:

a) $\displaystyle\lim_{x \to \infty} \frac{x^2 - 1}{2x^2 - x - 1}$;

b) $\displaystyle\lim_{x \to \infty} \frac{(x-1)(x-2)(x-3)(x-4)(x-5)}{(5x-1)^5}$;

c) $\displaystyle\lim_{x \to \infty} \frac{(2x-3)^{20}(3x+2)^{30}}{(2x+1)^{50}}$.

3. Prove that

a) $\lim_{x \to -\infty} (\sqrt{x^2 + x} - x) = +\infty$;

b) $\lim_{x \to +\infty} (\sqrt{x^2 + x} - x) = \frac{1}{2}$.

11. Limits of Sequences. The Greatest Lower Bound Property

In the case of a sequence $\{y_n\}$, i.e., a function with domain $X = \{1, 2, \ldots, n, \ldots\}$, Definition 3, p. 49 takes the following particularly simple form (with $E = X$):[6]

DEFINITION 1. *Given a finite point c,*[7] *the sequence* $\{y_n\}$ *is said to approach c (or converge to c) as n approaches* ∞ *if, given any* $\varepsilon > 0$, *there exists an integer* $N = N(\varepsilon) > 0$ *such that*

$$|y_n - c| < \varepsilon$$

if $n > N$. *This fact is expressed by writing* $y_n \to c$ *as* $n \to \infty$ *or*

$$\lim_{n \to \infty} y_n = c.$$

Example. The sequences with general terms

$$y_n = \frac{1}{n}, \quad y_n = \frac{(-1)^n}{n}, \quad y_n = \begin{cases} \dfrac{1}{n} & \text{if } n \text{ is odd,} \\ -\dfrac{1}{n^2} & \text{if } n \text{ is even} \end{cases}$$

all converge to zero. In the first two cases, the sequence $\{|y_n|\}$ is strictly decreasing, but not in the third case (compare y_{10} with y_{99}, say).

6. There is no need to say "relative to E" or "$x \in E$," since writing $x = n$, $f(x) = y_n$ already conveys this information. Moreover, in the case of a sequence, it is pointless to distinguish between $n \to \infty$ and $n \to +\infty$.

7. Here we exclude the case $c = \infty$, corresponding to a divergent sequence (see Definition 2 below).

DEFINITION 2. *A sequence $\{y_n\}$ is said to be convergent if it approaches a finite limit as $n \to \infty$. Otherwise $\{y_n\}$ is said to be divergent.*

THEOREM 2.15. *The sequence $\{y_n\}$ converges to c as $n \to \infty$ if and only if every neighborhood of c contains all but a finite number of terms of $\{y_n\}$.*

Proof. If $y_n \to c$ as $n \to \infty$, then, given any neighborhood $\mathscr{N}: |y - c| < \varepsilon$,[8] there is an integer $N = N(\varepsilon) > 0$ that such $y_n \in \mathscr{N}$ if $n > N$, and hence only finitely many terms of $\{y_n\}$ lie outside $\mathscr{N}$. Conversely, given any neighborhood $\mathscr{N}$: $|y - c| < \varepsilon$, suppose $\mathscr{N}$ contains all but a finite number of terms of $\{y_n\}$, and let N be the largest subscript of the terms of $\{y_n\}$ which lie outside $\mathscr{N}$. Then $n > N$ implies $y_n \in \mathscr{N}$, i.e., $|y_n - c| < \varepsilon$, and hence $y_n \to c$ as $n \to \infty$.

COROLLARY. *The limit of a convergent sequence $\{y_n\}$ is unique.*

Proof. If

$$\lim_{n\to\infty} y_n = c, \quad \lim_{n\to\infty} y_n = c',$$

where $c \neq c'$, let $\mathscr{N}$ and $\mathscr{N}'$ be nonoverlapping neighborhoods of c and c', respectively. Then $\mathscr{N}$ contains all but a finite number of terms of $\{y_n\}$, and so does $\mathscr{N}'$, which is obviously impossible. For another proof, see Theorem 2.1.

We are now in a position to prove a basic property of the real number system, whose importance can hardly be exaggerated. According to Sec. 5, a bounded set of real numbers E need not have a minimum or a maximum (see Example 3, p. 22). However, as we now show, E must have a greatest lower bound $\inf E$ and a least upper bound $\sup E$ (recall Definition 3, p. 22). The crucial difference is that $\inf E$ and $\sup E$ (unlike $\min E$ and $\max E$) need not belong to E itself.

THEOREM 2.16 *(Greatest lower bound property). If a set E is bounded from below, then E has a greatest lower bound* $\inf E$.

Proof. According to Remark 2, p. 23, the theorem is proved if $\min E$ exists. Thus suppose $\min E$ does not exist, and let m be

8. $\mathscr{N}: |y-c| < \varepsilon$ is shorthand for the set $\mathscr{N}$ of all y such that $|y - c| < \varepsilon$.

a lower bound of E. Then $m < x$ for all $x \in E$ (why?). Choose a point $x' \in E$ and consider the finite set of integers

$$[m],\ [m] + 1,\ \ldots,\ [x'],\ [x'] + 1, \tag{2.12}$$

where $[x]$ denotes the integral part of x, as in Example 5, p. 18. Let A be the largest number in the set (2.12) such that $A < x$ for all $x \in E$. Such a number exists, since $[m] < x$ for all $x \in E$ while $[x'] + 1 > x'$. Obviously $[m] \leqslant A < [x'] + 1$, and moreover the half-open interval $(A, A + 1]$ must contain at least one element of E, by the definition of A. If $A + 1 \in E$, then there must be an element of E smaller than $A + 1$, since min E does not exist. Therefore the *open* interval $(A, A + 1)$ contains elements of E, and hence there is an element $x_0 \in E$ such that

$$A < x_0 < A + 1.$$

Next consider the set of decimal expansions

$$A = A.0,\ \ A.1,\ \ \ldots,\ \ A.9,\ \ A + \frac{10}{10} = 1 + 1, \tag{2.13}$$

and let $y_1 = A.a_1$ be the largest number in (2.13) such that $y_1 < x$ for all $x \in E$. Such a number exists, since $A < x$ for all $x \in E$ while $A + 1 > x_0$. Just as before, there is an element $x_1 \in E$ such that

$$y_1 < x_1 < y_1 + \frac{1}{10}.$$

Applying the same argument to the decimal expansions

$$A.a_1 0,\ \ A.a_1 1,\ \ \ldots,\ \ A.a_1 9,\ \ A.a_1 + \frac{10}{10^2},$$

we find numbers $y_2 = A.a_1 a_2$ and $x_2 \in E$ such that $y_2 < x$ for all $x \in E$ and

$$y_2 < x_2 < y_2 + \frac{1}{10^2}.$$

Continuing this process indefinitely, we find a sequence $\{y_n\} = \{A.a_1 a_2 \cdots a_n\}$ and another sequence $\{x_n\}$, $x_n \in E$ such that

$y_n < x$ for all $x \in E$, while

$$y_n < x_n < y_n + \frac{1}{10^n}$$

or equivalently

$$0 < x_n - y_n < \frac{1}{10^n}. \tag{2.14}$$

Now let λ be the real number represented by the infinite decimal $A.a_1a_2 \ldots a_n \ldots$ (see Sec. 5). Clearly

$$\lambda - y_n < \frac{1}{10^n},$$

and hence

$$\lambda = \lim_{n\to\infty} y_n.$$

Since $y_n < x$ for all $x \in E$, it follows that $\lambda \leqslant x$ for all $x \in E$ (see the remark on p. 41), i.e., λ is a lower bound of E. Moreover, there can be no lower bound of E larger than λ, since if there were such a lower bound λ', then by (2.14), we could find a number in the sequence $\{x_n\}$ less than λ', since

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} [y_n + (x_n - y_n)] = \lim_{n\to\infty} y_n + \lim_{n\to\infty} (x_n - y_n)$$
$$= \lambda + 0 = \lambda.$$

Therefore

$$\lambda = \inf E,$$

i.e., λ is the required greatest lower bound of the set E, and the theorem is proved.

COROLLARY 1 *(Least upper bound property). If a set E is bounded from above, then E has a least upper bound* sup *E.*

Proof. Consider the set $-E$ obtained by replacing every element of E by its negative, and then note that

$$\sup E = -\inf(-E).$$

COROLLARY 2. *If a set E is bounded, then E has a greatest lower bound and a least upper bound.*

PROBLEMS

1. Which of the sequences with the following general terms have limits:

a) $y_n = \dfrac{1}{2^n}$; b) $y_n = \dfrac{n}{n+1}$; c) $y_n = \begin{cases} 1 & \text{for even } n, \\ \dfrac{1}{n} & \text{for odd } n; \end{cases}$

d) $y_n = \dfrac{1}{n} \cos \dfrac{n\pi}{2}$; e) $y_n = (-1)^n \dfrac{1}{n}$; f) $y_n = \dfrac{n+1}{n^2+2}$;

g) $y_n = \dfrac{1}{n - (-1)^n}$; h) $y_n = n - (-1)^n$; i) $y_n = n[1 - (-1)^n]$?

2. Find the limit of the sequence with general term

a) $y_n = \dfrac{1}{2n} + \dfrac{2n}{3n+1}$; b) $y_n = \dfrac{1}{3n} \sin n^2$;

c) $y_n = \dfrac{1}{2n} \cos n^3 - \dfrac{3n}{6n+1}$; d) $y_n = \dfrac{3n^2+2}{4n^2-1}$;

e) $y_n = \dfrac{(n+1)(n+2)(n-1)}{n^4+2n+3}$; f) $y_n = \dfrac{1+2+\cdots+n}{n^2}$.

3. Let $E_1 + E_2$ be the set of all sums $x + y$ with $x \in E_1$, $y \in E_2$. Prove that

$$\inf(E_1 + E_2) = \inf E_1 + \inf E_2,$$

$$\sup(E_1 + E_2) = \sup E_1 + \sup E_2.$$

4. Let E be the range of the function

$$f(x) = \frac{1}{x^2+1}$$

defined for all real x. Find $\max E$, $\min E$, $\sup E$ and $\inf E$.

12. The Bolzano-Weierstrass Theorem. The Cauchy Convergence Criterion

DEFINITION 1. *A finite point c is said to be a limit point of a sequence $\{y_n\}$ if every neighborhood of c contains infinitely many terms of $\{y_n\}$.*

It follows at once from Theorem 2.15 that the limit of a convergent sequence $\{y_n\}$ is a limit point of $\{y_n\}$. A divergent series can also have limit points, as shown by the example $\{y_n\} = \{(-1)^n\}$, which has two limit points -1 and 1. Definition closely resembles the related definition of the limit point of a set (Definition 1, p. 37). However, a point can be a limit point of a sequence without being a limit point of the set consisting of the terms of the sequence. Thus, although -1 and 1 are both limit points of $\{y_n\} = \{(-1)^n\}$, neither is a limit point of the set $\{-1, 1\}$ (recall from p. 37 that a finite set cannot have a limit point).

It is easy to construct sequences with no limit points at all. For example, the sequence of positive integers $\{n\}$ has no limit points, since no neighborhood of length 1 or smaller can contain more than one integer. However, as we now show, this case can only occur if the sequence is unbounded:

THEOREM 2.17 *(Bolzano-Weierstrass theorem). Every bounded sequence $\{y_n\}$ has a limit point.*

Proof. Since $\{y_n\}$ is bounded, there is a closed interval $[a, b]$ containing all the terms of $\{y_n\}$. Let E be the set of all real numbers x such that $y_n > x$ for only finitely many terms of $\{y_n\}$, or for no terms at all. Since $\{y_n\}$ is bounded from above, E is nonempty. Moreover, E is bounded from below by a, since $x < a$ implies $y_n > x$ for all (i.e., infinitely many) terms of $\{y_n\}$ and hence $x \in E$,[9] or equivalently $x \notin E$ implies $x \geqslant a$. Now let

$$\lambda = \inf E,$$

9. The symbol $\notin$ means "does not belong to."

where the existence of λ follows from Theorem 2.16. Then λ is a limit point of $\{y_n\}$. In fact, given any $\varepsilon > 0$, consider the neighborhood

$$\lambda - \varepsilon < x < \lambda + \varepsilon. \tag{2.15}$$

Since $\lambda - \varepsilon$ cannot belong to E, the inequality $y_n > \lambda - \varepsilon$ holds for infinitely many terms of the sequence $\{y_n\}$. Moreover, there are points of E to the left of $\lambda + \varepsilon$. Let x_1 be such a point. Then the inequality $y_n > x_1$ holds for only finitely many terms of $\{y_n\}$, and hence the same is true of the inequality $y_n \geqslant \lambda + \varepsilon$. But we have just seen that the inequality $y_n > \lambda - \varepsilon$ holds for infinitely many terms of $\{y_n\}$. Therefore infinitely many terms of $\{y_n\}$ must lie in the neighborhood (2.15). Since $\varepsilon > 0$ is arbitrary, λ is a limit point of $\{y_n\}$, as asserted.

Remark. The number λ found in the proof of Theorem 2.17 is the largest limit point of $\{y_n\}$. In fact, if $\lambda' > \lambda$, let

$$\varepsilon = \frac{\lambda' - \lambda}{2} > 0.$$

In the course of the proof, it was shown that

$$y_n \geqslant \lambda + \varepsilon = \lambda' - \varepsilon$$

holds for only finitely many terms of $\{y_n\}$. Therefore λ' is not a limit point of $\{y_n\}$.

DEFINITION 2. *Let $\{n_k\}$ be a sequence of positive integers, indexed by k and arranged in increasing order ($n_1 < n_2 < \cdots < n_k < \cdots$). Then, given any sequence $\{y_n\}$, the new sequence $\{y_{n_k}\}$ consisting of terms of $\{y_n\}$ is called a subsequence of $\{y_n\}$.*[10]

If $\{y_n\}$ converges to a limit c, then obviously so does every subsequence $\{y_{n_k}\}$. However, a subsequence of $\{y_n\}$ can converge even if $\{y_n\}$ itself diverges. For example, the sequence $\{y_n\} = \{(-1)^n\}$ diverges, but the subsequence $\{y_{2n}\}$ converges to 1 while $\{y_{2n+1}\}$ converges to -1. Clearly, if a subsequence $\{y_{n_k}\}$

10. Note that $\{y_{n_k}\}$ is itself indexed by k.

converges to the limit c, then c must be a limit point of the original sequence $\{y_n\}$. Moreover, the converse is also true, in the following sense:

THEOREM 2.18. *If c is a limit point of the sequence $\{y_n\}$, then $\{y_n\}$ contains a subsequence $\{y_{n_k}\}$ converging to c.*

Proof. Every neighborhood of the point c contains infinitely many terms of the sequence $\{y_n\}$. Let y_{n_1} be any term belonging to the neighborhood $|x - c| < 1$, let y_{n_2} be any term with a larger subscript belonging to the neighborhood $|x - c| < \frac{1}{2}$, and so on, so that

$$|y_{n_k} - c| < \frac{1}{k}, \quad n_1 < n_2 < \cdots < n_k$$

tor all k. Then obviously the subsequence $\{y_{n_k}\}$ converges to c, as required.

COROLLARY 1. *Every bounded sequence has a convergent subsequence.*

Proof. Use Theorem 2.17.[11]

COROLLARY 2. *If the sequence $\{y_n\}$ approaches c as $n \to \infty$, then c is the only limit point of $\{y_n\}$.*

THEOREM 2.19 *(Cauchy convergence criterion). The sequence $\{y_n\}$ is convergent if and only if, given any $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that*

$$|y_m - y_n| < \varepsilon \text{ whenever } m, n > N, \tag{2.16}$$

i.e., whenever both m and n exceed N.

Proof. First suppose $y_n \to c$ as $n \to \infty$. Then there exists a positive integer N such that $m, n > N$ implies

$$|y_m - c| < \frac{\varepsilon}{2}, \quad |y_n - c| < \frac{\varepsilon}{2}.$$

11. Thus Corollary 1 is essentially another version of the Bolzano-Weierstrass theorem.

Therefore

$$|y_m - y_n| = |(y_m - c) + (c - y_n)| \leqslant |y_m - c| + |y_n - c|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $m, n > N$, which agrees with (2.16). Conversely, suppose (2.16) holds. Then choosing $\varepsilon = 1$ (say) and $m_0 > N(1)$, we find that all but finitely many terms of $\{y_n\}$ lie in the neighborhood $|x - y_{m_0}| < 1$. Therefore the sequence $\{y_n\}$ is bounded, and it follows from the Bolzano-Weierstrass theorem that $\{y_n\}$ has a limit point c. Now let $N = N(\varepsilon/2)$ and choose $m_1 > N$ such that

$$|y_{m_1} - c| < \frac{\varepsilon}{2}$$

(such an integer m_1 exists by Theorem 2.18). Then

$$|y_n - c| = |(y_n - y_{m_1}) + (y_{m_1} - c)|$$
$$< |y_n - y_{m_1}| + |y_{m_1} - c| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $n > N$, i.e., $\{y_n\}$ is convergent (to the limit c) and the proof is complete.

PROBLEMS

1. Examine the limit points of the sequence

$$a, b, a^2, b^2, \ldots, a^n, b^n, \ldots$$

for various values of a and b.

2. Give an example of an unbounded sequence with a limit point.

3. Prove that the function $f(x)$ approaches a limit as $x \to x_0$ if and only if given any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that $0 < |x' - x_0| < \delta$ and $0 < |x'' - x_0| < \delta$ imply $|f(x') - f(x'')| < \varepsilon$.

Comment. This is the natural generalization of the Cauchy convergence criterion (Theorem 2.19).

4. Verify that the criterion of the preceding problem is satisfied by the function

$$f(x) = x \sin \frac{1}{x}$$

at the point $x = 0$, but not by the function

$$g(x) = \cos \frac{1}{x}.$$

5. Use the Cauchy convergence criterion to prove the convergence of the sequence with general term

$$y_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$$

and the divergence of the sequence with general term

$$y_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

13. Limits of Monotonic Functions. The Function a^x and the Number e

In general, there is no reason for a function to have a limit at any given point of its domain. However, in the case of monotonic functions (see p. 18), the existence of one-sided limits is easily proved:

THEOREM 2.20. *Let $f(x)$ be a bounded monotonic function with domain X, and let E be a subset of X with x_0 as a limit point such that E lies on one side of x_0.*[12] *Then the limit*

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x) \tag{2.17}$$

exists.

12. I.e., either $x \leqslant x_0$ for every $x \in E$, or $x \geqslant x_0$ for every $x \in E$.

Proof. To be explicit, suppose E lies to the right of x_0 and suppose $f(x)$ is increasing. Since $f(x)$ is bounded on E, it follows from Theorem 2.16 that the greatest lower bound

$$\lambda = \inf_{x \in E} f(x)$$

exists. It is not hard to see that λ is precisely the quantity (2.17). In fact, given any $\varepsilon > 0$, let $x_1 \in E$, $x_1 \neq x_0$ be such that

$$\lambda \leqslant f(x_1) < \lambda + \varepsilon, \text{ i.e., } 0 \leqslant f(x_1) - \lambda < \varepsilon$$

(such a point x_1 exists by the definition of λ). Since E lies to the right of x_0, we have $x_1 > x_0$ and hence $\delta = x_1 - x_0 > 0$. Let x be any point of E lying in the interval $(x_0, x_0 + \delta)$. Then $f(x) \leqslant f(x_1)$ since $f(x)$ is increasing, and $f(x) \geqslant \lambda$ by the definition of λ. Therefore

$$|f(x) - \lambda| = f(x) - \lambda \leqslant f(x_1) - \lambda < \varepsilon$$

provided that

$$x \in E, \quad 0 < |x - x_0| = x - x_0 < \delta.$$

In other words, the quantity (2.17) exists (and equals λ), as asserted.

In the case where $f(x)$ is *decreasing*, the same method of proof shows that

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x) = \sup_{x \in E} f(x).$$

Similarly, in the case where E lies to the *left* of x_0, it is easily seen that

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x) = \sup_{x \in E} f(x)$$

if $f(x)$ is increasing, while

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x) = \inf_{x \in E} f(x)$$

if $f(x)$ is decreasing.

COROLLARY 1. *If $f(x)$ is defined and increasing in a closed interval $[a, b]$, then the left and right-hand limits $f(x_0-)$ and $f(x_0+)$ exist at every point $x_0 \in (a, b)$, and*

$$f(x_0-) \leqslant f(x_0) \leqslant f(x_0+).$$

Moreover, $f(a+)$ and $f(b-)$ exist, and

$$f(a) \leqslant f(a+), \quad f(b-) \leqslant f(b).$$

Proof. If $x_0 \in (a, b)$, then $f(x_0-)$ and $f(x_0+)$ both exist, and $x < x_0$ implies $f(x) \leqslant f(x_0)$ while $x > x_0$ implies $f(x) \geqslant f(x_0)$. Therefore

$$f(x_0-) = \lim_{\substack{x \to x_0 \\ x \in [a,b] \\ x < x_0}} f(x) \leqslant f(x_0) \leqslant \lim_{\substack{x \to x_0 \\ x \in [a,b] \\ x > x_0}} f(x) = f(x_0+),$$

as asserted. The case of the end points is treated similarly.

COROLLARY 2. *A bounded monotonic sequence is convergent.*

Proof. Here $E = X$ is the set of all positive integers, and E lies to the left of its limit point ∞ (more exactly $+\infty$).

Example 1. Find the limit of the sequence $\{y_n\}$ where

$$y_n = q^n, \quad 0 < q < 1.$$

Solution. Since

$$y_{n+1} = q^{n+1} = q \cdot q^n = qy_n < y_n,$$

the sequence $\{y_n\}$ is decreasing. Moreover, $\{y_n\}$ is obviously bounded from above by q (the first term of the sequence) and from below by zero. It follows from Corollary 2 that $\{y_n\}$ converges to a limit c. To find c, we note that $y_{n+1} = qy_n$ implies

$$c = \lim_{n \to \infty} y_{n+1} = q \lim_{n \to \infty} y_n = qc,$$

and hence $c = 0$.

Example 2. Find the limit of the sequence $\{y_n\}$ where

$$y_n = a^{1/2^n}, \quad a > 1.$$

Solution. Since

$$y_{n+1} = a^{1/2^{n+1}} = a^{1/2 \cdot 2^n} = \sqrt{a^{1/2^n}} = \sqrt{y_n},$$

the sequence $\{y_n\}$ is decreasing. Moreover, $\{y_n\}$ is bounded from above by $\sqrt{a}$ (the first term of the sequence) and from below by 0, and hence $\{y_n\}$ approaches a limit c as $n \to \infty$. To find c, we note that $y_n = y_{n+1}^2$ implies

$$c = \lim_{n \to \infty} y_n = \lim_{n \to \infty} y_{n+1}^2 = c^2,$$

and hence $c = 0$ or $c = 1$. But every term of $\{y_n\}$ is greater than 1 (since $a > 1$). Therefore $c = 0$ is impossible and consequently $c = 1$.

The reader will recall from elementary mathematics that the function a^x $(a > 0)$, called the *exponential to the base a,* is defined on the set R of all rational numbers by the formulas

$$a^0 = 1, \quad a' = a, \quad a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}}, \quad a^{m/n} = (\sqrt[n]{a})^m,$$

$$a^{-n} = \frac{1}{a^n}, \quad a^{-m/n} = \frac{1}{a^{m/n}} \quad (m, n > 0).$$

If $a > 1$, then a^x is (strictly) increasing on R. In fact, there is no loss of generality in assuming that two rational numbers have the same denominator, say n, and then $m/n < m'/n$ implies

$$a^{m/n} = (\sqrt[n]{a})^m < (\sqrt[n]{a})^{m'} = a^{m'/n},$$

since $m < m'$ and $\sqrt[n]{a} > 1$ if $a > 1$. It is also easy to see that a^x satisfies the *addition theorem*

$$a^{x+x'} = a^x a^{x'} \quad (x, x' \in R). \tag{2.18}$$

We now extend the definition of a^x to the case of arbitrary real x:

THEOREM 2.21. *The limit*

$$\lim_{\substack{x \to x_0 \\ x \in R}} a^x \quad (a > 1) \tag{2.19}$$

exists for all real x_0, and equals a^{x_0} if x_0 is rational.

Proof. Since a^x is increasing on R and bounded in a neighborhood of any real x_0, the two limits

$$\lambda = \lim_{\substack{x \to x_0- \\ x \in R}} a^x, \quad \lambda' = \lim_{\substack{x \to x_0+ \\ x \in R}} a^x$$

exist, by Theorem 2.20. Let $\{r_n\}$ be any sequence of rational numbers approaching x_0 from the left, and let $\{r_n'\}$ be any sequence of rational numbers approaching x_0 from the right. Then, according to Theorem 2.8,

$$\lambda = \lim_{n \to \infty} a^{r_n}, \quad \lambda' = \lim_{n \to \infty} a^{r_n'},$$

and hence by (2.18),

$$\lambda' - \lambda = \lim_{n \to \infty} a^{r_n}(a^{r_n' - r_n} - 1) = \lim_{n \to \infty} a^{r_n} \lim_{n \to \infty} (a^{r_n' - r_n} - 1),$$

where $\{r_n' - r_n\}$ is a sequence approaching zero from the right. But $\{1/2^n\}$ is such a sequence, and hence, according to Example 2, above,

$$\lim_{n \to \infty} a^{r_n' - r_n} = \lim_{n \to \infty} a^{1/2^n} = 1.$$

It follows that $\lambda = \lambda'$, i.e.,

$$\lim_{\substack{x \to x_0- \\ x \in R}} a^x = \lim_{\substack{x \to x_0+ \\ x \in R}} a^x.$$

Therefore the limit (2.19) exists, again by Theorem 2.8. If x_0 is rational, then

$$\lim_{\substack{x \to x_0 \\ x \in R}} a^x = \lim_{\substack{x \to x_0 \\ x \in R}} a^{x_0} a^{x - x_0} = a^{x_0} \lim_{\substack{y \to 0 \\ y \in R}} a^y = a^{x_0} \cdot 1 = a^{x_0},$$

as asserted.

DEFINITION 1. *If x_0 is irrational, then a^{x_0} is defined as the limit* (2.19).

THEOREM 2.22. *The function a^x $(a > 1)$, defined for all real x, is increasing and satisfies the addition theorem* (2.18).

Proof. Given two real numbers x_1 and x_2 such that $x_1 < x_2$, let r_1 and r_2 be two rational numbers such that $x_1 < r_1 < r_2 < x_2$. Then $x < x_1$ (x rational) implies $a^x < a^{r_1}$, while $x > x_2$ implies $a^x > a^{r_2}$. It follows that

$$a^{x_1} = \lim_{\substack{x \to x_1 - \\ x \in R}} a^x \leqslant a^{r_1} < a^{r_2} \leqslant \lim_{\substack{x \to x_2 + \\ x \in R}} = a^{x_2},$$

i.e., a^x is (strictly) increasing. To verify (2.18) for arbitrary real numbers, let $\{r_n\}$ be a sequence of rational numbers converging to x and let $\{r_n'\}$ be a sequence of rational numbers converging to x'. Then $r_n + r_n' \to x + x'$ as $n \to \infty$, and hence

$$a^{x+x'} = \lim_{n \to \infty} a^{r_n + r_n'} = \lim_{n \to \infty} a^{r_n} a^{r_n'} = \lim_{n \to \infty} a^{r_n} \lim_{n \to \infty} a^{r_n'} = a^x a^{x'}.$$

Remark. If $0 < a < 1$, then

$$a^x = \frac{1}{(1/a)^x} = \frac{1}{a'^x},$$

where $a' = 1/a > 1$. Therefore the addition theorem continues to hold in this case, but a^x is now a *decreasing* function.

Finally we turn our attention to some limits which will play an important role later:

THEOREM 2.23. *The sequence $\{y_n\}$ with general term*

$$y_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

is convergent.

Proof. First we show that if $x \geqslant -1$, then

$$(1 + x)^m \geqslant 1 + mx \tag{2.20}$$

for all $m = 1, 2, \ldots$, a result known as *Bernoulli's inequality*. In fact, suppose (2.20) holds for $m = k$, so that

$$(1 + x)^k \geqslant 1 + kx. \tag{2.21}$$

Multiplying (2.21) by $1 + x$ ($\geqslant 0$), we obtain

$$\begin{aligned}(1 + x)^{k+1} &= (1 + kx)(1 + x) = 1 + (k + 1)x + kx^2 \\ &\geqslant 1 + (k + 1)x,\end{aligned}$$

i.e., (2.20) holds for $m = k + 1$, and hence, by mathematical induction for every positive integer m, since (2.20) obviously holds for $m = 1$.

Now, according to (2.20),

$$\left(\frac{1 + \dfrac{1}{n}}{1 + \dfrac{1}{n+1}}\right)^{n+1} = \left(1 + \frac{1}{n(n+2)}\right)^{n+1} \geqslant 1 + \frac{n+1}{n(n+2)}$$

$$> 1 + \frac{n+1}{(n+1)^2} = 1 + \frac{1}{n+1}.$$

Therefore

$$\frac{y_n}{y_{n+1}} = \frac{\left(1 + \dfrac{1}{n}\right)^{n+1}}{\left(1 + \dfrac{1}{n+1}\right)^{n+2}}$$

$$= \left(\frac{1 + \dfrac{1}{n}}{1 + \dfrac{1}{n+1}}\right)^{n+1} \frac{1}{1 + \dfrac{1}{n+1}}$$

$$> \frac{1 + \dfrac{1}{n+1}}{1 + \dfrac{1}{n+1}} = 1,$$

i.e., the sequence $\{y_n\}$ is (strictly) decreasing. Moreover, $\{y_n\}$ is bounded from above by its first term and from below by zero. It follows from Theorem 2.20, Corollary 2 that $\{y_n\}$ is convergent.

DEFINITION 2. *The limit*

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n+1}$$

is denoted by e. Logarithms to the base e are called natural logarithms, and the natural logarithm of x is denoted by ln x.[13]

Remark 1. Natural logarithms turn out to be so important in higher mathematics that the number e itself is called the *base of the natural logarithms.*

Remark 2. Since

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$
$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\cdot 1 = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n},$$

we might just as well have defined e as

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}.$$

Remark 3. To estimate e, we note that

$$y_5 = (1+\tfrac{1}{5})^6 < 3,$$

while

$$y_n = \left(1+\frac{1}{n}\right)^{n+1} \geqslant 1 + \frac{n+1}{n} > 2.$$

It follows that $2 < e < 3$. A more exact calculation shows that $e = 2.7182818\ldots$[14]

THEOREM 2.24. *The limit*

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}$$

exists and equals e.

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13. Logarithms to an arbitrary base a are written as $\log_a x$, and hence $\log_e x = \ln x$. Moreover, we write $\log_{10} x = \log x$, dropping the subscript if $a = 10$.

14. Based on the approximation

$$e \approx 2 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{10!} + \frac{1}{11!}$$

(not proved here), where $n! = 1\cdot 2\cdots n$.

Proof. It makes sense to talk about the limit of the function

$$\left(1 + \frac{1}{x}\right)^x$$

as $x \to \infty$, since

$$1 + \frac{1}{x} > 0$$

if $x < -1$ or $x > 0$ (recall that a^x has only been defined for $a > 0$). First let $x \to +\infty$. Clearly

$$\left(1 + \frac{1}{[x] + 1}\right)^{[x]} < \left(1 + \frac{1}{x}\right)^x < \left(1 + \frac{1}{[x]}\right)^{[x]+1} \tag{2.22}$$

where $[x]$ is the integral part of x (see p. 18). If $x \to +\infty$, then $[x] \to +\infty$ and the right-hand side of (2.22) approaches e by Theorem 2.23. But the left-hand side of (2.22) also approaches e, since

$$\left(1 + \frac{1}{[x] + 1}\right)^{[x]} = \frac{\left(1 + \dfrac{1}{[x] + 1}\right)^{[x]+2}}{\left(1 + \dfrac{1}{[x] + 1}\right)^2} \to e \cdot 1 = e$$

as $[x] \to +\infty$. Therefore

$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e \tag{2.23}$$

by Theorem 2.9.

Next let $x \to -\infty$. Writing

$$\left(1 + \frac{1}{x}\right)^x = \left(\frac{x + 1}{x}\right)^x = \left(\frac{x}{x + 1}\right)^{-x} = \left(1 + \frac{1}{-x - 1}\right)^{-x}$$

$$= \left(1 + \frac{1}{-x - 1}\right)^{-x-1} \left(1 + \frac{1}{-x - 1}\right)$$

$$= \left(1 + \frac{1}{y}\right)^y \left(1 + \frac{1}{y}\right),$$

we see that since $y = -x - 1 \to +\infty$ as $x \to -\infty$,

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to+\infty}\left(1+\frac{1}{y}\right)^y\left(1+\frac{1}{y}\right) = e\cdot 1 = e. \tag{2.24}$$

Together (2.23) and (2.24) imply

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e,$$

as required.

COROLLARY. *The limit*

$$\lim_{x\to 0}(1+x)^{1/x}$$

exists and equals e.

PROBLEMS

1. Use the Cauchy convergence criterion (Theorem 2.19) to prove the convergence of the sequence with general term

$$y_n = \frac{\sin 1}{2} + \frac{\sin 2}{2^2} + \cdots + \frac{\sin n}{2^n}.$$

2. Find the limit of the sequence with general term

$$y_n = \frac{a^n}{1+a^{2n}}.$$

3. Prove that the sequence

$$\sqrt{2},\ \sqrt{2}^{\sqrt{2}},\ \sqrt{2}^{\sqrt{2}^{\sqrt{2}}},\ \ldots,\ \sqrt{2}^{\overbrace{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot^{\sqrt{2}}}}}}}^{n\text{ times}}},\ \ldots$$

converges.

Hint. The sequence is increasing and bounded (why?).

4. Find

$$\lim_{n\to\infty}\frac{n!}{n^n},$$

where $n! = 1\cdot 2\cdots n$.

5. Prove that the sequence $\{y_n\}$ with general term

$$y_n = \left(1 + \frac{1}{n}\right)^n$$

is strictly increasing.

Comment. As noted on p. 69, $y_n \to e$ as $n \to \infty$.

6. Prove that

$$\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}$$

for every positive integer n.

7. Prove that

$$\ln(n+1) < 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

for every positive integer n.

Hint. Use the preceding problem.

CHAPTER 3

Continuity

4. Continuous Functions

After introducing the general concept of a function in Sec. 1, we described some simple function classes in Sec. 4 (even and odd functions, periodic and monotonic functions). Now that the concept of a limit is at our disposal, we are in a position to introduce another class of functions, of the greatest importance both in theory and in the applications.

DEFINITION 1. *A function $f(x)$ defined in a neighborhood of a point x_0 is said to be continuous at x_0 if $f(x)$ has a limit at x_0 and if this limit equals the value of $f(x)$ at x_0, i.e., if*

$$\lim_{x \to x_0} f(x) = f(x_0). \tag{3.1}$$

More exactly, $f(x)$ is said to be continuous at x_0 if, given any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon$$

if $|x - x_0| < \delta$.[1] *If* (3.1) *does not hold at a point x_0, then $f(x)$ is said to be discontinuous at x_0.*

Remark 1. The limit of a function $f(x)$ at a point x_0 may exist even though $f(x)$ is undefined at x_0, but $f(x)$ cannot be continuous at x_0 unless it is defined at x_0, since the very definition of continuity involves the value $f(x_0)$. By the same token, even if $f(x)$

1. Here we write $|x - x_0| < \delta$ instead of $0 < |x - x_0| < \delta$ as in Definition 1, p. 24, since obviously $|f(x_0) - f(x_0)| = 0 < \delta$ if $f(x_0)$ is known to exist.

is defined at x_0, its limit at x_0 does not depend on $f(x_0)$, but changing the value of $f(x)$ at x_0 has a profound effect on the continuity of $f(x)$ at x_0, since we thereby change the right-hand side of (3.1) without changing the left-hand side. On the other hand, changing the value of $f(x)$ at a point other than x_0 can have no effect on the continuity of $f(x)$ at x_0, which only involves values of $f(x)$ in the "immediate vicinity" of x_0, including the point x_0 itself (cf. Remark 4, p. 24).

Remark 2. The notion of continuity has already been anticipated in the remark on p. 26 and in the definition of the function a^x for irrational x (see Sec. 13). In fact, a^x was defined for irrational x by "continuous extension" of its values for rational x.

Given two values x and x_0 of the independent variable x, the difference

$$\Delta x = x - x_0$$

is called the *increment* of x at x_0. Correspondingly, the difference $f(x) - f(x_0)$ between the values of a function $f(x)$ at the points x and x_0 is called the *increment* of $f(x)$ at x_0, written

$$\Delta f(x_0) = f(x) - f(x_0).$$

In this language, the definition of continuity takes the following form: A function $f(x)$ is continuous at a point x_0 if its increment can be made less than any preassigned $\varepsilon > 0$ by making the increment of its argument sufficiently small, i.e., if

$$\lim_{\Delta x \to 0} [f(x) - f(x_0)] = \lim_{\Delta x \to 0} \Delta f(x_0) = 0.$$

Remark. Thus, intuitively speaking, a continuous function is one which "has no breaks" in the sense that its graph can be drawn without lifting pen from paper. It is easy to see why such functions are of great importance in science and technology, where the principle "nature does not make a jump" reigns supreme (at least in classical physics!). For example, common sense and actual experiment show that the positions of a moving body at two consecutive instants of time are arbitrarily close if the interval between the two times is sufficiently small.

THEOREM 3.1. *If $f(x)$ is continuous at x_0 and $f(x_0) > 0$, then $f(x) > 0$ in some neighborhood of x_0. Similarly, if $f(x)$ is continuous at x_0 and $f(x_0) < 0$, then $f(x) < 0$ in some neighborhood of x_0.*

Proof. If $f(x_0) > 0$, let $\varepsilon = \frac{1}{2}f(x_0)$. Then there exists a $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$|f(x) - f(x_0)| < \tfrac{1}{2}f(x_0)$$

and hence

$$f(x) > \tfrac{1}{2}f(x_0) > 0.$$

The case $f(x_0) < 0$ is treated similarly.

COROLLARY. *If $f(x)$ is continuous at x_0 and if $f(x)$ takes both positive and negative values in an arbitrary neighborhood of x_0, then $f(x_0) = 0$.*

As already noted, continuity at a point x_0 is a local notion, involving only the values of the function near x_0. Thus a function can be continuous at one point and discontinuous at others. For example, consider the function

$$f(x) = x\,[1 - 2D(x)],$$

where $D(x)$ is the Dirichlet function introduced on p. 8. Clearly $f(x)$ is continuous at $x_0 = 0$ since $f(0) = 0$ and

$$\lim_{x \to 0} x = 0,$$

while $1 - 2D(x)$ is bounded. On the other hand, $f(x)$ is discontinuous at every other point, and in fact $f(x)$ does not even approach a limit at any other point. To see this, let R be the set of all rational numbers and I the set of all irrational numbers. Then

$$\lim_{\substack{x \to x_0 \\ x \in R}} f(x) = \lim_{x \to x_0} x\,(1 - 2) = -x_0,$$

whereas

$$\lim_{\substack{x \to x_0 \\ x \in I}} f(x) = \lim_{x \to x_0} x\,(1 - 0) = x_0.$$

But these two limits differ, and hence

$$\lim_{x \to x_0} f(x)$$

fails to exist, unless $x_0 = 0$.

DEFINITION 2. *A function $f(x)$ defined on a set E is said to be continuous on (or in) E iff $f(x)$ is continuous at every point $x_0 \in E$, where the limit of $f(x)$ at x_0 is understood to be relative to the set E.*

Example 1. The function x^2 is continuous in the open unit interval $(0,1)$, since obviously

$$\lim_{x \to x_0} x^2 = \lim_{x \to x_0} x \cdot \lim_{x \to x_0} x = x_0^2$$

for every real x_0, in particular for all $x_0 \in (0,1)$. However, the function $x\,[1 - 2D(x)]$ is not continuous in $(0,1)$.

Example 2. The function $f(x) = x^2$ is continuous in the closed unit interval $[0,1]$ since $f(x)$ is continuous in $(0,1)$ and

$$\lim_{x \to 1-} f(x) = f(1) = 1, \quad \lim_{x \to 0+} f(x) = f(0) = 0.$$

Example 3. The function $f(x) \equiv \text{const}$ is continuous on the whole real line.

Next we investigate the continuity of combinations of continuous functions:

THEOREM 3.2. *If $f(x)$ and $g(x)$ are continuous at x_0, then so are the functions $f(x) \pm g(x)$ and $f(x)\,g(x)$. Moreover, if $g(x_0) \neq 0$, the function $f(x)/g(x)$ is also continuous at x_0.*[2]

Proof. The theorem is an immediate consequence of Theorems 2.5, 2.6 and 2.7, on algebraic properties of limits.

COROLLARY. *If $f_1(x), \ldots, f_n(x)$ are all continuous at x_0, then so are the functions $f_1(x) \pm \cdots \pm f_n(x)$ and $f_1(x) \cdots f_n(x)$.*

Proof. Apply Theorem 3.2 repeatedly.

Remark. Thus certain algebraic combinations of continuous functions are themselves continuous. Interestingly enough,

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2. It obviously makes no sense to talk about continuity of $f(x)/g(x)$ at points where $g(x)$ vanishes, since $f(x)/g(x)$ is not defined at such points.

algebraic combinations of discontinuous functions may also be continuous. For example, as already noted, the function

$$f(x) = x\,[1 - 2D(x)]$$

is discontinuous everywhere except at the origin. However, the function

$$\varphi(x) = [f(x)]^2 = x^2\,[1 - 2D(x)]^2$$

is continuous everywhere, and in fact $\varphi(x) = x^2$ since $\varphi(x) = x^2\,(1 - 2)^2 = x^2$ if x is rational while $\varphi(x) = x^2\,(1 - 0)^2 = x^2$ if x is irrational. We note in passing that there are at least five solutions of the equation

$$[f(x)]^2 - x^2 = 0, \tag{3.2}$$

namely

$$f(x) = x,\ -x,\ |x|,\ -|x|,\ x\,[1 - 2D(x)].$$

In fact, there are actually infinitely many solutions of (3.2)!

DEFINITION 3. *Let $f(y)$ be a function with domain Y and let $g(x)$ be a function with domain X and range $E \subset Y$. Then by the composite function $f[g(x)]$, defined on X, is meant the function which associates the number $f[g(x_0)]$ with every fixed $x_0 \subset X$.*

Remark. Composite functions like $f\{g[h(x)]\}$ are defined similarly. Note that the range of each function (except the first) must be a subset of the domain of the function written on its left.

THEOREM 3.3. *If $g(x)$ is continuous at x_0 and if $f(y)$ is continuous at $y_0 = f(x_0)$, then $f[g(x)]$ is continuous at x_0.*

Proof. Given any $\varepsilon > 0$, there is a $\delta_1 > 0$ such that $|y - y_0| < \delta_1$ implies

$$|f(y) - f(y_0)| < \varepsilon,$$

and then a $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$|y - y_0| = |g(x) - g(x_0)| < \delta_1.$$

Therefore $|x - x_0| < \delta$ implies

$$|f[g(x)] - f[g(x_0)]| = |f(y) - f(y_0)| < \varepsilon,$$

i.e.,

$$\lim_{x \to x_0} f[g(x)] = f[g(x_0)],$$

as required.

PROBLEMS

1. Use an "ε, δ-argument" to show that the function $f(x) = x^2$ is continuous at $x = 5$. In particular, fill in the following table:

ε	1	0.1	0.01	0.001
δ				

2. Suppose that given any sufficiently small $\delta > 0$, there exists an $\varepsilon = \varepsilon(\delta) > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$. Does this mean that $f(x)$ is continuous at x_0? If not, what does it mean?

3. Prove that if $f(x)$ is continuous on any given set, then so is $|f(x)|$.

4. Which of the following functions are continuous at the origin:

a) $f(x) = |x|$;

b) $f(x) = \dfrac{|x|}{x}$ if $x \neq 0, f(0) = 0$;

c) $f(x) = \sin \dfrac{1}{x}$ if $x \neq 0, f(0) = 0$;

d) $f(x) = x \sin \dfrac{1}{x}$ if $x \neq 0, f(0) = 0$?

5. The function

$$f(x) = \frac{1 - \cos x}{x^2}$$

is defined for all $x \neq 0$. What value should $f(x)$ be assigned at $x = 0$ to make it continuous at $x = 0$?

6. Investigate the continuity of the functions $f[g(x)]$ and $g[f(x)]$ if $f(x) = \operatorname{sgn} x$ and

a) $g(x) = 1 + x^2$; b) $g(x) = x\,(1 - x^2)$; c) $g(x) = 1 + [x]$.

(The function $\operatorname{sgn} x$ is defined in Prob. 6, p. 11.)

7. Suppose $f(x)$ is continuous at x_0 while $g(x)$ is discontinuous at x_0. Must $f(x) + g(x)$ be discontinuous at x_0? How about $f(x)\,g(x)$?

15. One-Sided Continuity. Classification of Discontinuities

DEFINITION 1. *A function $f(x)$ is said to be continuous from the left at a point x_0 if $f(x)$ is defined in some interval $(x_0 - \delta, x_0]$ and if*

$$f(x_0-) = \lim_{x \to x_0-} f(x) = f(x_0).$$

DEFINITION 2. *A function $f(x)$ is said to be continuous from the right at a point x_0 if $f(x)$ is defined in some interval $[x_0, x_0 + \delta)$ and if*

$$f(x_0) = \lim_{x \to x_0+} f(x) = f(x_0).$$

Remark. Thus a function $f(x)$ is continuous in a closed interval $[a, b]$ if and only if it is continuous in the open interval (a, b), continuous from the left at b and continuous from the right at a (cf. Definition 2 and Example 2, both on p. 76).

DEFINITION 3. *A function $f(x)$ is said to have a discontinuity of the first kind at x_0 if $f(x_0-)$ and $f(x_0+)$ both exist but at least one of these limits differs from $f(x_0)$.*

Example 1. Figure 11a shows a function whose one-sided limits $f(x_0-)$ and $f(x_0+)$ coincide but differ from $f(x_0)$, the value of the function at x_0. Therefore, although $f(x)$ has a limit at x_0, it has a discontinuity of the first kind at x_0. However, the discontinuity can be removed by changing the value of $f(x)$ at the single point x_0, i.e., by assigning $f(x_0)$ the new value $f(x_0-) = f(x_0+)$, and for this reason, the discontinuity is said to be *removable.* Figure 11b shows a function $f(x)$ whose one-sided limits at x_0 differ. In this case, the discontinuity is again of the first kind, but not removable since $f(x)$ has no limit at x_0.

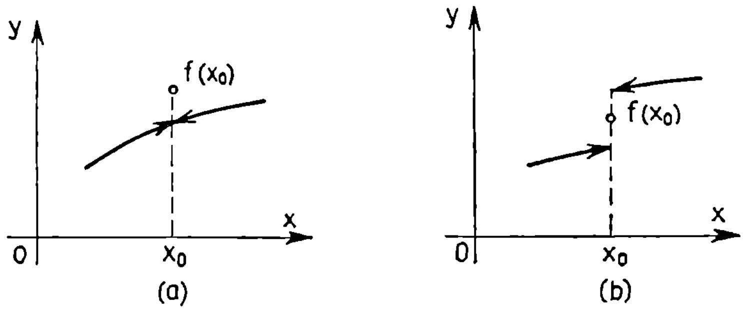

Fig. 11

DEFINITION 4. *A function $f(x)$ is said to have a discontinuity of the second kind at x_0 if at least one of the limits $f(x_0-)$ and $f(x_0+)$ fails to exist or is infinite.*

Example 2. The function $1/x$ has a discontinuity of the second kind at $x = 0$, since

$$\lim_{x\to 0-} \frac{1}{x} = -\infty, \quad \lim_{x\to 0+} \frac{1}{x} = +\infty.$$

Example 3. The Dirichlet function $D(x)$ is bounded but has no one-sided limits at any point x_0. Therefore $D(x)$ has a discontinuity of the second kind at every point.

Example 4. The function $\cos(\pi/x)$, $x \neq 0$ plotted in Figure 12 has no right-hand (or left-hand) limit at the origin. In fact, if $x_n = \dfrac{1}{2n}$, then

$$\cos\frac{\pi}{x_n} = \cos 2n\pi = 1,$$

for all $n = \pm 1, \pm 2, \ldots$, while if $x_n' = \dfrac{1}{2n+1}$, then

$$\cos\frac{\pi}{x_n'} = \cos(2n+1)\pi = -1,$$

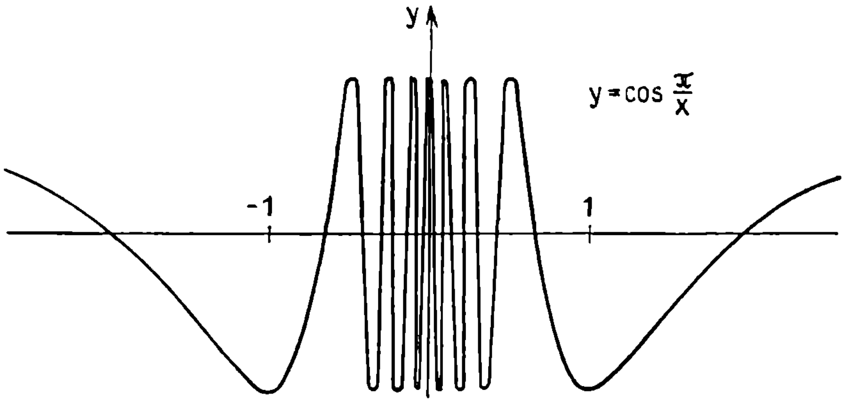

Fig. 12

for all $n = \pm 1, \pm 2, \ldots$, and hence

$$\lim_{x \to 0+} \cos \frac{\pi}{x}$$

fails to exist (why?). Therefore $\cos(\pi/x)$ has a discontinuity of the second kind at the origin.

THEOREM 3.4. *Let $f(x)$ be monotonic in a closed interval $[a, b]$. Then $f(x)$ is continuous in $[a, b]$ if it takes all values between $f(a)$ and $f(b)$. Otherwise, the discontinuities of $f(x)$ are all of the first kind.*

Proof. If $f(x)$ is increasing, Theorem 2.20, Corollary 1 guarantees the existence of $f(x_0-)$ and $f(x_0+)$, and hence the discontinuities of $f(x)$, if any, must all be of the first kind. By the same theorem,

$$f(x_0-) \leqslant f(x_0) \leqslant f(x_0+)$$

if $x_0 \in (a, b)$, and hence

$$f(x_0-) < f(x_0+)$$

if x_0 is a point of discontinuity. (There are obvious modifications in the case where $x_0 = a$ or $x_0 = b$.) But then $f(x)$ cannot take all values in the interval $[f(x_0-), f(x_0+)] \subset [f(a), f(b)]$, but only the value $f(x_0)$, since obviously

$$f(x) \leqslant f(x_0-) \text{ if } x < x_0,$$

$$f(x) \geqslant f(x_0+) \text{ if } x > x_0.$$

The case of decreasing $f(x)$ is treated similarly.

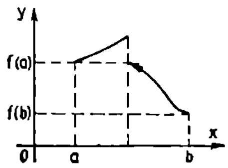

Fig. 13

Remark. A nonmonotonic function $f(x)$ can take all values between $f(a)$ and $f(b)$ without being continuous. In fact, the discontinuous function shown in Figure 13 takes all values between $f(a)$ and $f(b)$.

PROBLEMS

1. Find and classify the discontinuities of each of the following functions:

a) $y = [x]$; b) $y = \{x\}$; c) $y = [x] + [-x]$;

d) $y = \dfrac{|x|}{x}$; e) $y = x + \dfrac{1}{x}$; f) $y = \dfrac{1}{x^2}$;

g) $y = \dfrac{1}{x^2 - 1}$; h) $y = e^{1/x}$.

2. Find and classify the discontinuities of each of the following functions:

a) $y = \dfrac{x}{(1 + x)^2}$; b) $y = \dfrac{1 + x}{1 + x^3}$; c) $y = \dfrac{x^2 - 1}{x^3 - 3x + 2}$;

d) $y = \dfrac{\dfrac{1}{x} - \dfrac{1}{x + 1}}{\dfrac{1}{x - 1} - \dfrac{1}{x}}$; e) $y = e^{x + \frac{1}{x}}$; f) $y = \sqrt{\dfrac{1 - \cos \pi x}{4 - x^2}}$;

g) $y = \cos^2 \left(\dfrac{1}{x}\right)$; h) $y = \operatorname{sgn}\left(\sin \dfrac{\pi}{x}\right)$; i) $y = \dfrac{1}{1 - e^{\frac{x}{1-x}}}$.

3. How many functions $y = f(x)$ satisfy the equation

$$x^2 + y^2 = 4?$$

Among these functions find two that are continuous in the interval $[-2, 2]$. Find the function which is negative in $[-1, 1]$

and nonnegative for all other admissible x. Draw the graph of this function and indicate its points of discontinuity. Are these discontinuities removable?

16. The Intermediate Value Theorem. Absolute Extrema

DEFINITION 1. *A sequence of closed intervals* $\Delta_n = [a_n, b_n]$ *is said to be nested if* $\Delta_{n+1} \subset \Delta_n$ *for all* $n = 1, 2, \ldots$ *and if* $b_n - a_n \to 0$ *as* $n \to \infty$.

THEOREM 3.5. *Let* $\{\Delta_n\}$ *be a nested sequence of closed intervals. Then there is one and only one point* c *contained in every* Δ_n.

Proof. First we prove the uniqueness of c. Suppose there is another point c' ($c' > c$ say) contained in every Δ_n. Then

$$a_n \leqslant c < c' \leqslant b_n,$$

$$b_n - a_n \geqslant c' - c = \lambda > 0,$$

and hence

$$\lim_{n\to\infty} (b_n - a_n) \geqslant \lambda > 0,$$

contrary to hypothesis. This contradiction shows that there is at most one point c in every Δ_n.

To prove the existence of such a point c, we note that

$$a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n,$$

$$b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n,$$

$$a_n < b_n \leqslant b_1,$$

so that $\{a_n\}$ is bounded and increasing, while $\{b_n\}$ is decreasing. Therefore $\{a_n\}$ is convergent, by Theorem 2.20, Corollary 2, with limit

$$c = \lim_{n\to\infty} a_n,$$

where

$$a_n \leqslant c, \tag{3.3}$$

since $\{a_n\}$ is increasing.[3] Moreover

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} (b_n - a_n) = c + 0 = c,$$

and hence

$$b_n \geqslant c, \tag{3.4}$$

since $\{b_n\}$ is decreasing. Together (3.3) and (3.4) imply

$$a_n \leqslant c \leqslant b_n, \quad \text{i.e., } c \in \varDelta_n \quad (n = 1, 2, \ldots),$$

as required.

COROLLARY. *Let $\{\varDelta_n\}$ and c be the same as in Theorem 3.5. Then, given any $\delta > 0$, every $\varDelta_n$ is contained in the neighborhood $(c - \delta, c + \delta)$ starting from some sufficiently large value of n.*

THEOREM 3.6. *If $f(x)$ is continuous in $[a, b]$ and if $f(a)$ and $f(b)$ have opposite signs, then $f(x)$ vanishes at some point $c \in (a, b)$.*

Proof. Assuming (without loss of generality) that $f(a) < 0 < f(b)$, let d be the midpoint of the interval $[a, b]$. If $f(d) = 0$, the theorem is proved. Otherwise, let $\varDelta_1$ denote the interval $[a, d]$ if $f(d) > 0$ and the interval $[d, b]$ if $f(d) < 0$, and write $\varDelta_1 = [a_1, b_1]$. Then $f(a_1) < 0, f(b_1) > 0$ in any event. Let d_1 be the midpoint of $\varDelta_1$. If $f(d_1) = 0$, the theorem is proved. Otherwise, of the two halves of $\varDelta_1$ with common end point d_1, choose the interval $\varDelta_2 = [a_2, b_2]$ such that $f(a_2) < 0, f(b_2) > 0$. Continuing this construction indefinitely, we either eventually obtain an interval at whose midpoint $f(x)$ vanishes, thereby proving the theorem, or else a nested sequence of closed intervals $\varDelta_n = [a_n, b_n]$ such that $f(a_n) < 0, f(b_n) > 0$.[4] In the latter case,

---

3. Note that if $a_n \leqslant a_{n+p}$ for all $n, p = 1, 2, \ldots$, then

$$a_n \leqslant \lim_{p\to\infty} a_{n+p} = c.$$

4. Note that $\varDelta_{n+1} \subset \varDelta_n$ and

$$b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1}) = \frac{1}{2^n}(b - a)$$

for all $n = 1, 2, \ldots$ ($a_0 = a, b_0 = b$).

according to Theorem 3.5 and its corollary, there is a unique point c contained in every Δ_n, and moreover, given any $\delta > 0$, the neighborhood $(c - \delta,\ c + \delta)$ contains every Δ_n starting from some sufficiently large value of n. But then $f(x)$ takes both positive and negative values in every neighborhood of c, since $f(a_n) < 0$, $f(b_n) > 0$. Therefore $f(c) = 0$ by the corollary to Theorem 3.1, and the theorem is proved.

Remark. Theorem 3.6 can also be proved by using the greatest lower bound property (see Theorem 2.16), but the present proof is better, since it gives an explicit method for estimating roots of the equation $f(x) = 0$. In fact, if the original interval $[a, b]$ is of unit length, the seventh subdivision gives an interval Δ_7 of length

$$\frac{1}{2^7} < \frac{1}{100},$$

i.e., any point of Δ_7 differs from a root of $f(x) = 0$ by less than $1/100$.

THEOREM 3.7 *(Intermediate value theorem). If $f(x)$ is continuous in $[a, b]$, then $f(x)$ takes every value between $f(a)$ and $f(b)$ at some point of $[a, b]$.*

Proof. The function $\varphi(x) = f(x) - C$, where C is any number between $f(a)$ and $f(b)$, is continuous, being the difference between two continuous functions. Moreover, $\varphi(a)$ and $\varphi(b)$ have opposite signs. Therefore, by Theorem 3.6, there is a point $c \in (a, b)$ such that $\varphi(c) = f(c) - C = 0$, i.e., such that $f(c) = C$, as required.

Remark. Let $f(x)$ be monotonic in a closed interval $[a, b]$. Then, according to Theorems 3.4 and 3.7, $f(x)$ is continuous if and only if $f(x)$ takes all values between $f(a)$ and $f(b)$. This assertion depends on $f(x)$ being monotonic (see the remark on p. 82).

DEFINITION 2. *A function $f(x)$ defined on a set E is said to have an absolute minimum (in E) at the point $x_0 \in E$ if $f(x_0) \leqslant f(x)$ for every $x \in E$. Similarly, $f(x)$ is said to have an absolute maximum (in E) at the point $x_0 \in E$ if $f(x_0) \geqslant f(x)$ for every $x \in E$. The value $f(x_0)$ is called the absolute minimum in the first case and the* *absolute maximum in the second case. The term absolute extremum refers to either an absolute minimum or an absolute maximum.*

DEFINITION 3. *By $f(E)$ is meant the set of all numbers $f(x)$ such that $x \in E$.*

THEOREM 3.8. *The function $f(x)$ has an absolute minimum in E if and only if $f(E)$ has a minimum. Similarly, $f(x)$ has an absolute maximum in E if and only if $f(E)$ has a maximum.*

Proof. The theorem is an immediate consequence of Definitions 2 and 3 above and Definitions 1 and 2, p. 22.

Remark. Obviously, $f(x)$ is bounded from below in E if $f(x)$ has an absolute minimum in E, bounded from above in E if $f(x)$ has an absolute maximum in E, and bounded in E if $f(x)$ has both an absolute minimum and an absolute maximum in E.

Example 1. If $f(x)$ is increasing in $[a, b]$, then $f(x)$ has an absolute minimum at $x = a$ and an absolute maximum at $x = b$.

Example 2. The function

$$f(x) = \tan x, \quad x \in E = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

is unbounded both from below and from above in E, and hence has neither an absolute minimum nor an absolute maximum in E.

Example 3. The function

$$f(x) = x, \quad x \in E = (0, 1)$$

is bounded in E, but has neither an absolute minimum nor an absolute maximum in E.

The last two examples show that a function defined in an open interval need not have absolute extrema. However, as we shall see in a moment, the situation is different if the interval is closed.

THEOREM 3.9. *If $f(x)$ is continuous in $[a, b]$, then $f(x)$ is bounded in $[a, b]$.*

Proof. Suppose to the contrary that $f(x)$ is continuous but unbounded in $[a, b]$. Let d be the midpoint of $[a, b]$, and consider the intervals $[a, d]$ and $[d, b]$. Clearly, $f(x)$ must be unbounded in at least one of these subintervals, since otherwise $f(x)$ would be bounded in $[a, b]$, contrary to hypothesis. Denoting this subinterval by $\Delta_1 = [a_1, b_1]$, let d_1 be the midpoint of Δ_1. Then, of the two halves of Δ_1 with common end point d_1 choose an interval $[a_2, b_2]$ in which $f(x)$ is unbounded (such an interval exists for the same reason as before). Continuing this construction indefinitely, we obtain a nested sequence of closed intervals $\Delta_n = [a_n, b_n]$ such that $f(x)$ is unbounded in every Δ_n.[5] Let $c \in [a,b]$ be the unique point contained in every Δ_n (see Theorem 3.5). Then $f(x)$ is continuous at c, since it is continuous at every point of $[a, b]$. Hence there is a neighborhood $(c - \delta, c + \delta)$ in which $f(x)$ is bounded,[6] since $f(x)$ can be made arbitrarily close to $f(c)$ by choosing δ small enough. But then $f(x)$ is bounded in every Δ_n contained in $(c - \delta, c + \delta)$, i.e., in every Δ_n starting from some sufficiently large value of n (see the corollary to Theorem 3.5). This contradicts the way in which the sequence $\{\Delta_n\}$ is selected, thereby proving the theorem.

THEOREM 3.10. *If $f(x)$ is continuous in $[a, b]$, then $f(x)$ has both an absolute minimum and an absolute maximum in $[a, b]$, equal to*[7]

$$m = \inf_{a \leqslant x \leqslant b} f(x) \quad \textit{and} \quad M = \sup_{a \leqslant x \leqslant b} f(x),$$

respectively.

Proof. By Theorem 3.9, $f(x)$ is bounded and hence

$$m = \inf_{a \leqslant x \leqslant b} f(x)$$

exists, by the greatest lower bound property (Theorem 2.16).

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5. The construction closely resembles that given in the proof of Theorem 3.6. In particular, the intervals $\Delta_n = [a_n, b_n]$ have the same properties as in footnote 4, p. 84.

6. The neighborhood $(c - \delta, c + \delta)$ is replaced by $[c, c + \delta)$ if $c = a$ and by $(c - \delta, c (c - \delta, c]$ if $c = b$.

7. By $\inf_{a \leq x \leq b} f(x)$ is meant the greatest lower bound of all numbers $f(x)$ with $a \leqslant x \leqslant b$. Similarly, $\sup_{a \leq x \leq b} f(x)$ is the least upper bound of all numbers $f(x)$ with $a \leqslant x \leqslant b$. We can also write $m = \inf f([a, b])$ and $M = \sup f([a, b])$, recalling Definition 3, p. 86.

By definition, $f(x) \geqslant m$ for every $x \in [a, b]$. Suppose there is no point $x_1 \in [a, b]$ such that $f(x_1) = m$. Then $f(x) > m$ for every $x \in [a, b]$, and the function

$$\varphi(x) = \frac{1}{f(x) - m}$$

is continuous in $[a, b]$, being the ratio of two functions continuous in $[a, b]$ (the denominator cannot vanish). By Theorem 3.9 again, $\varphi(x)$ is bounded in $[a, b]$ and hence

$$0 < \varphi(x) = \frac{1}{f(x) - m} \leqslant C < \infty,$$

i.e.,

$$f(x) \geqslant m + \frac{1}{C}.$$

But then

$$m = \inf_{a \leqslant x \leqslant b} f(x) \geqslant m + \frac{1}{C} > m,$$

which is impossible. Hence there must be a point $x_1 \in [a, b]$ such that $f(x_1) = m$, and then $f(x)$ has an absolute minimum at x_1. Similarly, let x_2 be the point in $[a, b]$ at which $-f(x)$ has an absolute minimum. Then $f(x)$ has an absolute maximum at x_2, equal to

$$-\inf_{a \leqslant x \leqslant b} [-f(x)] = \sup_{a \leqslant x \leqslant b} f(x),$$

and the theorem is proved.

PROBLEMS

1. Prove that given any a with $|a| < 1$, there is one and only one point x_0 in the interval $(-\pi/2, \pi/2)$ such that $\sin x_0 = a$.

2. Prove that the equation

$$\sin x - x + 1 = 0$$

has at least one solution in the interval $(2\pi/3, 3\pi/4)$.

3. Use the greatest lower bound property to prove Theorem 3.6.

4. Prove that the function

$$f(x) = \cos\frac{1}{x} \quad (x \neq 0)$$

takes every value between -1 and $+1$ (including ± 1) in any neighborhood of $x = 0$ (and hence cannot be continuous at $x = 0$).

5. Is the function $f(x) = \{x\}$ (the fractional part of x) bounded in $(-\infty, +\infty)$?

6. Give an example of a function which is discontinuous in a closed interval $[a, b]$ and has absolute extrema in $[a, b]$.

17. Inverse Functions

Let $y = f(x)$ be a function with domain X and range Y, and let y_0 belong to Y. Then the equation $f(x) = y_0$ has at least one root $x_0 \in X$, and possibly several, as in the case shown in Figure 14 (where there are three roots).

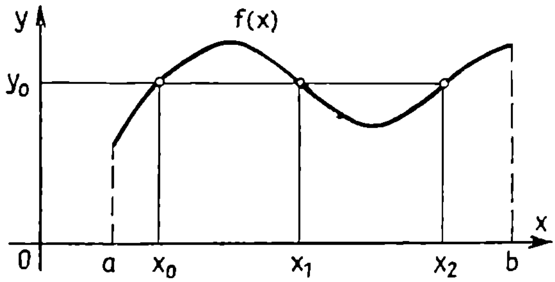

Fig. 14

Definition 1. *A function $y = f(x)$ with domain X and range Y is said to be one-to-one if the equation $f(x) = y_0$ has only one root for every $y_0 \in Y$.*

Definition 2. *Let $y = f(x)$ be a one-to-one function with domain X and range Y, and let $x = \varphi(y)$ be the function whose* *value at the point* $y_0 \in Y$ *is the unique solution of the equation* $f(x) = y_0$. *Then* $x = \varphi(y)$ *is called the inverse (function) of* $y = f(x)$.

Clearly, if $f(x)$ is a one-to-one function with domain X and range Y, then its inverse $\varphi(y)$ is a one-to-one function with domain Y and range X, and moreover

$$f[\varphi(y)] \equiv y, \quad \varphi[f(x)] \equiv x.$$

Remark. One can also introduce the concept of a *multiple-valued* function, which assigns *several* values to at least one point in its domain. In this sense, the function $f(x)$ shown in Figure 14 has a multiple-valued inverse $\varphi(x)$, and we can no longer write

$$\varphi[f(x)] \equiv x,$$

since for example

$$\varphi[f(x_0)] = \varphi(y_0) = \{x_0, x_1, x_2\},$$

where $\{x_0, x_1, x_2\}$ is the set containing the three points x_0, x_1 and x_2. Although multiple-valued functions play an important role in higher mathematics, they will not be considered further here. Thus we shall henceforth be concerned exclusively with ordinary (i.e., "single-valued") functions, as defined in Sec. 1.

THEOREM 3.11. *Let* $f(x)$ *be strictly increasing in the closed interval* $[a, b]$. *Then its inverse* $\varphi(y)$ *exists and is strictly increasing in* $[f(a), f(b)]$. *Moreover, if* $f(x)$ *is continuous, so is* $\varphi(y)$.

Proof. The existence of $\varphi(y)$ follows from the fact that $f(x)$ is one-to-one. Let y_1, y_2 $(y_1 < y_2)$ be two points in the domain of $\varphi(y)$, and let x_1, x_2 be the corresponding points in $[a, b]$ such that $y_1 = f(x_1)$, $y_2 = f(x_2)$. Then $x_1 \geqslant x_2$ is impossible, since this would imply $y_1 = f(x_1) \geqslant f(x_2) = y_2$, contrary to hypothesis. Therefore $x_1 < x_2$ if $y_1 < y_2$, i.e., $\varphi(y)$ is strictly increasing. If $f(x)$ is continuous, as well as strictly increasing, then according to Theorem 3.7, the range of $f(x)$ and hence the domain of $\varphi(y)$ is the closed interval $[f(a), f(b)]$. But $\varphi(y)$ obviously takes every value between a and b, by construction, and hence $\varphi(y)$ is continuous, by Theorem 3.4.

Remark 1. It is easy to see that Theorem 3.11 remains true if the word "increasing" is replaced by "decreasing" and hence by "monotonic."

Remark 2. The inverse of a strictly monotonic function is easily found by reflecting the graph of $f(x)$ in the line bisecting the first quadrant, as shown in Figure 15 where $A = f(a)$, $B = f(b)$.

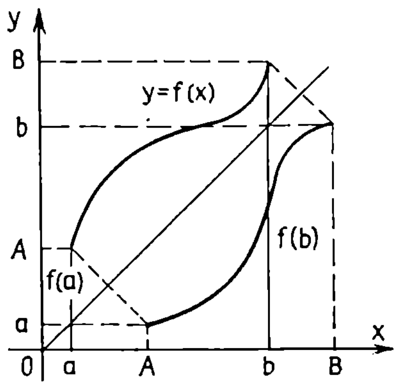

Fig. 15

PROBLEMS

1. Find the inverse of each of the following functions:

a) $y = 3x$; b) $y = 5 - 2x$; c) $y = x^2 - 2$;

d) $y = \dfrac{1}{2 - x}$; e) $y = x^2 - 4x$; *f*) $y = x$; g) $y = \dfrac{1}{x}$;

h) $y = \sqrt{x}$; i) $y = \sqrt[3]{x + 1}$; j) $y = \sqrt[3]{x^2 + 1}$;

k) $y = 2^x - 1$.

2. Prove that the function

$$y = \frac{1 - x}{1 + x}$$

is its own inverse.

3. Find the inverse of the function

$$y = \begin{cases} x^2 - 4x + 6 & \text{if} \quad x \leqslant 2, \\ -x + 4 & \text{if} \quad x \geqslant 2. \end{cases}$$

4. Find the inverse of the function

$$y = x + [x].$$

18. Elementary Functions

We now discuss some particularly simple functions, repeatedly encountered in mathematical analysis.

1. Polynomials

By a *polynomial of degree n* is meant a function of the form

$$P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n \quad (a_0 \neq 0),$$

where $a_0, a_1, \ldots, a_n$ are real numbers.[8] Clearly $P(x)$ is defined and continuous for every real x, and is in fact obtained by repeated addition and multiplication of the continuous functions $f(x) \equiv \text{const}$ and $f(x) = x$.

2. Rational functions

By a *rational function* is meant a ratio of two polynomials

$$R(x) = \frac{P(x)}{Q(x)},$$

defined for every real x except the points where the denominator $Q(x)$ vanishes. Moreover, as the ratio of two continuous functions, $R(x)$ is continuous at every point of its domain of definition.

8. We allow $a_0 = 0$ if $n = 0$, i.e., the function $f(x) \equiv 0$ is classified as a (trivial) polynomial.

3. Irrational functions

By allowing the extraction of roots as well as addition, multiplication and division, we obtain *irrational functions*. For example, the function $f(x) = \sqrt[n]{x}$ (the inverse of x^n) is irrational.

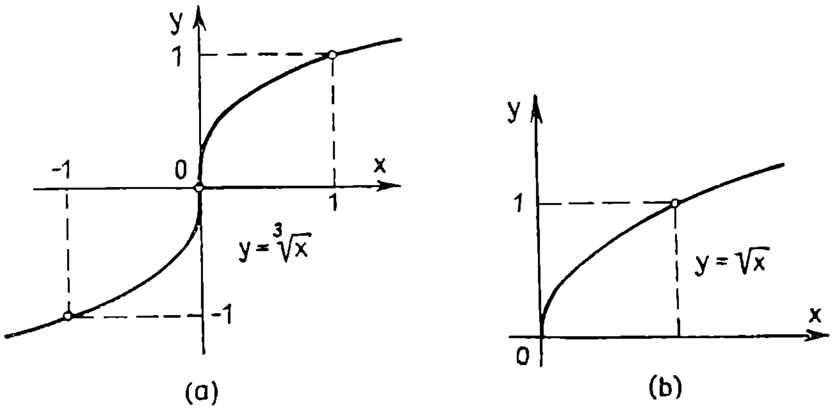

Fig. 16

If n is odd, x^n is strictly increasing and continuous in $(-\infty, +\infty)$ and hence the same is true of $\sqrt[n]{x}$, by Theorem 3.11. For example, consider the function $\sqrt[3]{x}$ shown in Figure 16a. If n is even, the function x^n is strictly increasing and continuous in $[0, +\infty)$, but not in $(-\infty, +\infty)$. Therefore $\sqrt[n]{x}$ is also strictly increasing and continuous in $[0, +\infty)$. The situation is shown in Figure 16b for the case $n = 2$.

4. Exponentials

The *exponential to the base a*, i.e., the function a^x $(a > 0)$, has already been defined for arbitrary x in Sec. 13, and the fact that a^x is continuous in $(-\infty, +\infty)$ follows at once from Theorem 2.21, p. 65 and Definition 1, p. 66. Obviously a^x is strictly increasing if $a > 1$, and then

$$\lim_{x\to-\infty} a^x = \lim_{n\to\infty} a^{-n} = \lim_{n\to\infty}\left(\frac{1}{a}\right)^n = \lim_{n\to\infty} q^n = 0,$$

where we have used Example 1, p. 64 and the fact that

$$0 < q = \frac{1}{a} < 1.$$

Moreover

$$\lim_{x \to +\infty} a^x = \lim_{y \to -\infty} \frac{1}{a^y} = +\infty,$$

and hence a^x $(a > 1)$ has the range $(0, +\infty)$. If $0 < a < 1$, the formula

$$a^x = \frac{1}{(1/a)^x}$$

implies that a^x is strictly decreasing, again with range $(0, +\infty)$. Figure 17 shows the function a^x for the two cases $a > 1$ and $a < 1$.

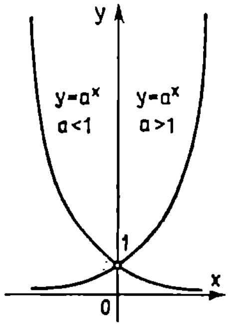

Fig. 17

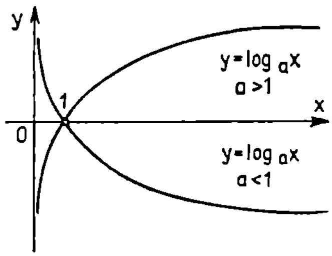

Fig. 18

5. Logarithms

By the *logarithm to the base a* $(a > 0)$, denoted by $\log_a x$, we mean the inverse of the function a^x. Since a^x has domain $(-\infty, +\infty)$ and range $(0, +\infty)$, $\log_a x$ has domain $(0, +\infty)$ and range $(-\infty, +\infty)$. Moreover, it follows from Theorem 3.11 and the corresponding properties of a^x that $\log_a x$ is strictly increasing and continuous if $a > 1$ and strictly decreasing and continuous if $a < 1$. The formula $a^0 = 1$ implies $\log_a 1 = 0$ and hence

$$\log_a x < 0 \quad \text{for} \quad x < 1,$$

$$\log_a x > 0 \quad \text{for} \quad x > 1$$

if $a > 1$, while

$$\log_a x > 0 \quad \text{for} \quad x < 1,$$

$$\log_a x < 0 \quad \text{for} \quad x > 1$$

if $a < 1$. Figure 18 shows the function $\log_a x$ for the two cases $a > 1$ and $a < 1$.

6. The function x^α

If α is a rational number, then x^α is a polynomial if α is a nonnegative integer, a rational function if α is a negative integer and an irrational function if α is a fraction. If α is an irrational number, then x^α is defined for all $x > 0$ by continuity, as in Sec. 13. Taking the logarithm of the formula

$$y = x^\alpha,$$

we find that

$$\log_a y = \alpha \log_a x,$$

and hence

$$y = x^\alpha = a^{\alpha \log_a x}. \tag{3.5}$$

Since $\alpha \log_a x$ is continuous with domain $(0, +\infty)$ and range $(-\infty, +\infty)$, while a^x is continuous with domain $(-\infty, +\infty)$ and range $(0, +\infty)$, it follows from Theorem 3.3 that the composite function (3.5) is continuous with domain and range $(0, +\infty)$.

7. Trigonometric functions

The basic trigonometric functions are

$$\sin x, \cos x, \tan x, \cot x.$$

As we know from Sec. 4, the function $\sin x$, defined for all x, is odd and periodic with period 2π. Moreover, according to Theorem 2.2,

$$\lim_{x\to 0} \sin x = \lim_{x\to 0} x \frac{\sin x}{x} = 0 \cdot 1 = 0 = \sin 0,$$

and hence $\sin x$ is continuous at the origin. To prove that $\sin x$ is continuous in $(-\infty, +\infty)$, let x_0 be an arbitrary real number.

Then

$$|\sin x - \sin x_0| = 2\left|\sin \frac{x - x_0}{2}\right|\left|\cos \frac{x + x_0}{2}\right|$$

$$\leqslant 2\left|\sin \frac{x - x_0}{2}\right|,$$

where, as just shown, the right-hand side approaches 0 as $x \to x_0$. But then

$$\lim_{x \to x_0} |\sin x - \sin x_0| = 0,$$

i.e.,

$$\lim_{x \to x_0} \sin x = \sin x_0,$$

so that $\sin x$ is continuous at x_0. The continuity of $\cos x$ in $(-\infty, +\infty)$ follows from the formula

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

and the continuity of the functions $\sin x$ and $\frac{\pi}{2} - x$. Therefore, by Theorem 3.2, the function

$$\tan x = \frac{\sin x}{\cos x}$$

is continuous everywhere except where $\cos x = 0$, i.e., except at the points $\frac{\pi}{2} \pm k\pi\,(k = 0, 1, 2, \ldots)$, while $\cot x$ is continuous everywhere except at the points $\pm k\pi\ (k = 0, 1, 2, \ldots)$.

8. Inverse trigonometric functions

Since $\sin x$ is strictly increasing and continuous in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, its inverse function $\arcsin x$ is strictly increasing and continuous in the interval $\left[-\sin\frac{\pi}{2}, \sin\frac{\pi}{2}\right] = [-1, 1]$, as shown in Figure 19a. Moreover, since $\cos x$ is strictly decreasing and continuous in $[0, \pi]$, its inverse arc cos x is strictly decreasing and continuous in $[\cos \pi, \cos 0] = [-1, 1]$, as shown in Figure 19b. The properties of arc tan x and arc cot x are found similarly, and are shown in Figures 19c and 19d.

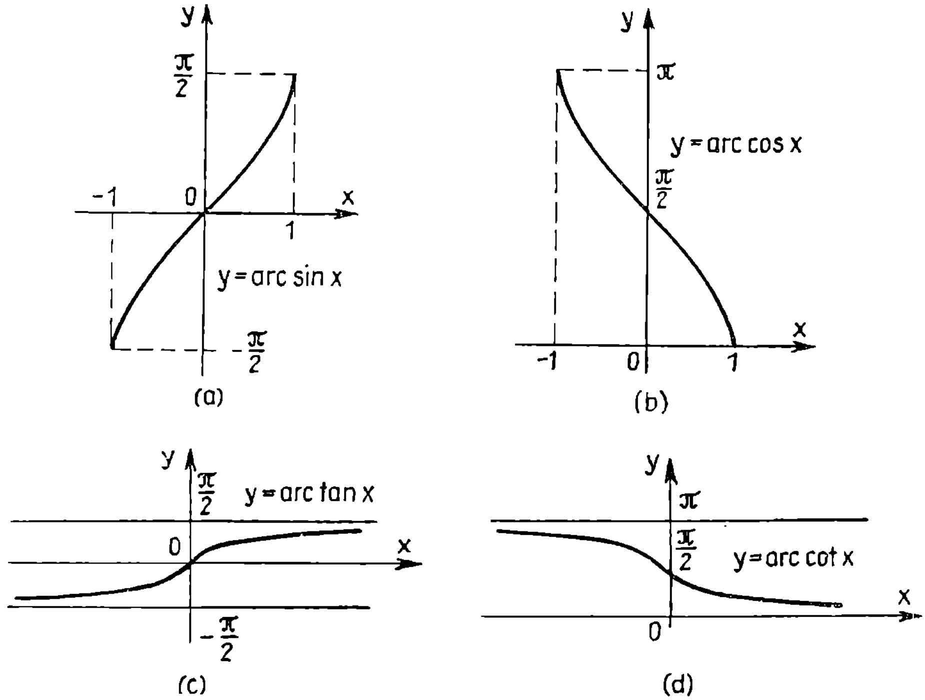

Fig. 19

9. More general elementary functions

Any function formed from the above functions by a finite number of algebraic operations (addition, subtraction, multiplication and division) and a finite number of compositions (i.e., formation of composite functions) will also be called *elementary*. For example, the functions

$$\sin^3 (x + 2 \text{ arc tan } x), \quad 6x + \sqrt{\log_a x},$$

$$\frac{e^x + e^{-x}}{2}, \frac{e^x - e^{-x}}{2}$$

are all elementary (the last two have the special names of *hyperbolic cosine* and *hyperbolic sine*). Using Theorems 3.2 and 3.3 and the continuity of the functions listed above, we can deduce the continuity of more general elementary functions (on suitable domains).

Example 1. Study the function $\sqrt{\sin x}$ and plot its graph.

Solution. Since $\sin x$ is periodic with period 2π, the same is true of $\sqrt{\sin x}$, i.e.,

$$\sqrt{\sin (x \pm 2k\pi)} = \sqrt{\sin x} \quad (k = 0, 1, 2, \ldots),$$

provided that $\sqrt{\sin x}$ exists. Thus to plot the graph of $\sqrt{\sin x}$, we need only examine its behavior in the interval $[0, 2\pi]$, or for that matter in the smaller interval $[0, \pi]$, since $\sin x$ is negative and hence $\sqrt{\sin x}$ fails to exist if $x \in (\pi, 2\pi)$. Moreover, $\sqrt{\sin x}$ is symmetric with respect to the line $x = \frac{1}{2}\pi$ since

$$\sin\left(\frac{\pi}{2} - x\right) = \sin\left(\frac{\pi}{2} + x\right),$$

and hence we can actually confine our attention to the interval $[0, \frac{1}{2}\pi]$. Calculating the values of $\sqrt{\sin x}$ at the four points 0, $\frac{1}{6}\pi$, $\frac{1}{3}\pi$ and $\frac{1}{2}\pi$, we obtain

$$\sqrt{\sin 0} = 0, \sqrt{\sin\frac{\pi}{6}} = \frac{\sqrt{2}}{2} \approx 0.7,$$

$$\sqrt{\sin\frac{\pi}{3}} = \sqrt{\frac{\sqrt{3}}{2}} \approx 0.9, \quad \sqrt{\sin\frac{\pi}{2}} = 1.$$

Joining the corresponding points by a "smooth" curve,[9] we obtain the graph shown in Figure 20. We then reflect this curve in the line $x = 2/\pi$ and use the periodicity to construct the rest of

9. The fact that the curve $y = \sqrt{\sin x}$ has no "breaks" follows at once from the continuity of $\sqrt{\sin x}$. It can also be shown (by a method beyond the scope of this book) that the curve has no "kinks."

the graph. The final result is the solid curve shown in Figure 21, where for purposes of comparison, the dashed curve shows the function sin x. Unlike our previous examples of periodic functions (e.g., sin x and tan x), the function $\sqrt{\sin x}$ fails to exist in infinitely many intervals.

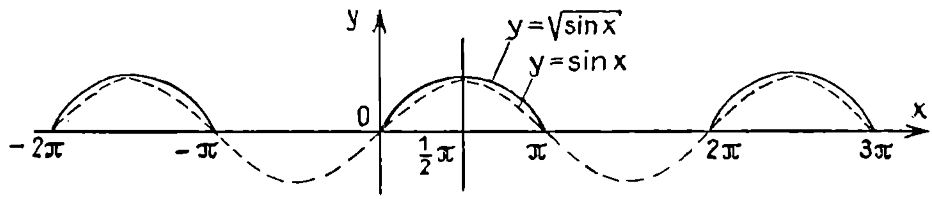

Fig. 21

Example 2. Study the function cos (sin x) *and plot its graph.*

Solution. Obviously, this function exists and is continuous for all x, and is periodic with period 2π. However, it also has the smaller period π, since

$$\cos [\sin (x + \pi)] = \cos (-\sin x) = \cos (\sin x). \quad (3.6)$$

Because of (3.6) and the evenness of the function, we need only consider its behavior in the interval $[0, \frac{1}{2}\pi]$. Calculating a few values of cos (sin x), we join the corresponding points by a smooth curve. We then reflect the resulting curve in the line $x = 0$ and use the periodicity, thereby obtaining the graph shown in Figure 22. Note that this function is positive for all x.

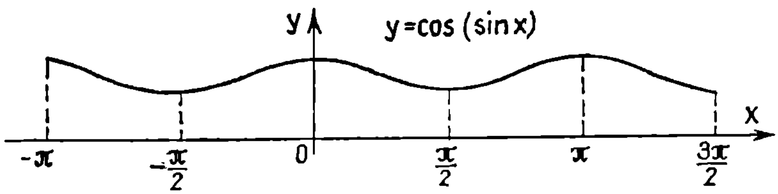

Fig. 22

PROBLEMS

1. Discuss the behavior of the polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n \quad (a_0 \neq 0)$$

as $x \to \pm\infty$ and $x \to 0$.

2. Find the set of all x such that

a) $\arcsin x + \arccos x = \dfrac{\pi}{2}$;

b) $\arcsin \sqrt{x} + \arccos \sqrt{x} = \dfrac{\pi}{2}$;

c) $\arccos \sqrt{1 - x^2} = \arcsin x$;

d) $\arccos \sqrt{1 - x^2} = -\arcsin x$.

3. Study the hyperbolic functions

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2},$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

drawing a graph of the function in each case. Prove that

a) $\cosh^2 x - \sinh^2 x = 1$;

b) $\sinh (x + y) = \sinh x \cosh y + \cosh x \sinh y$;

c) $\cosh (x + y) = \cosh x \cosh y + \sinh x \sinh y$;

d) $\tanh (x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$;

e) $\sinh 2x = 2 \sinh x \cosh y$;

f) $\cosh 2x = \cosh^2 x + \sinh^2 x$.

Compare these formulas with the corresponding formulas for trigonometric functions.

4. The inverses of the functions sinh x, cosh x and tanh x are denoted by arc sinh x, arc cosh x and arc tanh x, respectively. Prove that

a) $\operatorname{arc\,sinh} x = \ln\left(x + \sqrt{x^2 + 1}\right)$;

b) $\operatorname{arc\,cosh} x = \ln\left(x + \sqrt{x^2 - 1}\right)$;

c) $\operatorname{arc\,tanh} x = \dfrac{1}{2} \ln \dfrac{1 + x}{1 - x}$.

5. Study the function $y = \sin x^2$ and draw its graph.

19. Evaluation of Limits

Limits can often be found very simply by using the continuity of elementary functions. For example, the continuity of the functions $x^{\sqrt{2}}$ and $\sin e^x$ implies

$$\lim_{x \to 3} x^{\sqrt{2}} = 3^{\sqrt{2}}, \quad \lim_{x \to 0} \sin e^x = \sin e^0 = \sin 1.$$

These results would have been more difficult to obtain directly, without recourse to the theory developed earlier in this chapter.

We now use continuity to establish a further property of the function a^x:

THEOREM 3.12. *The function a^x $(a > 0)$ satisfies the relation*

$$a^{xy} = (a^x)^y = (a^y)^x. \tag{3.7}$$

Proof. For rational x and y, formula (3.7) is familiar from elementary mathematics. Suppose y is rational but x is irrational and let R be the set of all rational numbers. Then

$$a^{xy} = \lim_{\substack{t \to x \\ t \in R}} a^{ty} = \lim_{\substack{t \to x \\ t \in R}} (a^t)^y = \lim_{\substack{t \to x \\ t \in R}} (a^y)^t,$$

which implies (3.7) since the functions a^x and x^α are both continuous. If x and y are both irrational, then

$$a^{xy} = \lim_{\substack{t \to y \\ t \in R}} a^{xt} = \lim_{\substack{t \to y \\ t \in R}} (a^x)^t = \lim_{\substack{t \to y \\ t \in R}} (a^t)^x,$$

which again implies (3.7), for the same reason.

COROLLARY. *The relation*

$$\log_a x^y = y \log_a x \quad (a > 0) \tag{3.8}$$

holds for arbitrary real $x > 0$ and y.

Proof. Use the formulas

$$x^y = a^{\log_a x^y}, \quad x^y = (a^{\log_a x})^y = a^{y \log_a x}$$

and the fact that x^α is a one-to-one function.

Next we evaluate some particularly important limits:

THEOREM 3.13. *The following formulas hold:*

$$\lim_{x \to 0} \frac{\log_a (1 + x)}{x} = \log_a e, \tag{3.9}$$

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a, \tag{3.10}$$

$$\lim_{x \to 1} \frac{x^\alpha - 1}{x - 1} = \alpha. \tag{3.11}$$

Proof. To prove (3.9), we note that the function $\log_a x$ is continuous at the point $x = e$. Therefore

$$\begin{aligned}\lim_{x \to 0} \frac{\log_a (1 + x)}{x} &= \lim_{x \to 0} \frac{1}{x} \log_a (1 + x) \\ &= \lim_{x \to 0} \log_a (1 + x)^{1/x} \\ &= \log_a \lim_{x \to 0} (1 + x)^{1/x} = \log_a e\end{aligned}$$

(justify the next to the last step), where we have used (3.8) and the corollary to Theorem 2.24.

To prove (3.10), we write

$$y = a^x - 1, \ a^x = 1 + y, \ x = \log_a (1 + y)$$

and note that $y \to 0$ as $x \to 0$. It follows that

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \lim_{x\to 0} \frac{y}{\log_a (1 + y)} = \frac{1}{\log_a e} = \log_e a = \ln a,$$

where we have used (3.9) and the familiar fact that

$$\log_a b = \frac{1}{\log_b a}.$$

Finally, to prove (3.11), we note that

$$\frac{x^\alpha - 1}{x - 1} = \frac{e^{\alpha \ln x} - 1}{\alpha \ln x} \frac{\alpha \ln x}{x - 1} = \frac{e^y - 1}{y} \frac{\alpha \ln (1 + t)}{t},$$

where $y = \alpha \ln x$ and $t = x - 1$ both approach zero as $x \to 1$. Therefore, because of (3.9) and (3.10),

$$\lim_{x\to 1} \frac{x^\alpha - 1}{x - 1} = \lim_{y\to 0} \frac{e^y - 1}{y} \lim_{t\to 0} \frac{\alpha \ln (1 + t)}{t} = \alpha \ln e = \alpha.$$

Remark. Formula (3.11) can also be written in the form

$$\lim_{t\to 0} \frac{(l + t)^\alpha - 1}{t} = \alpha, \tag{3.12}$$

by setting $t = x - 1$.

Example 1. Find the limit

$$\lim_{x\to 1} \frac{\sqrt[5]{x} - 1}{\sqrt[3]{x} - 1}.$$

Solution. Dividing the numerator and denominator by $x - 1$ and using (3.11), we obtain

$$\lim_{x\to 1} \frac{\sqrt[5]{x} - 1}{\sqrt[3]{x} - 1} = \lim_{x\to 1} \frac{\dfrac{x^{1/5} - 1}{x - 1}}{\dfrac{x^{1/3} - 1}{x - 1}} = \frac{\dfrac{1}{5}}{\dfrac{1}{3}} = \frac{3}{5}.$$

Example 2. Find the limit

$$\lim_{x\to 0} \frac{2^x - 1}{\sqrt{1 + x} - 1}.$$

Solution. Dividing the numerator and denominator by x and using (3.10) and (3.12), we obtain

$$\lim_{x\to 0} \frac{2^x - 1}{\sqrt{1 + x} - 1} = \lim_{x\to 0} \frac{\dfrac{2^x - 1}{x}}{\dfrac{(1 + x)^{1/2} - 1}{x}} = \frac{\ln 2}{\dfrac{1}{2}} = \ln 4.$$

Example 3. Find the limit

$$\lim_{x\to 0} \frac{\log (1 + x)}{10^x - 1}$$

($\log x = \log_{10} x$).

Solution. As before, we have

$$\lim_{x\to 0} \frac{\dfrac{\log (1 + x)}{x}}{\dfrac{10^x - 1}{x}} = \frac{\log e}{\ln 10} = (\log e)^2.$$

PROBLEMS

1. Prove that

$$\lim_{x\to a} \frac{\ln x - \ln a}{x - a} = \frac{1}{a}$$

if $a > 0$.

2. Find the following limits:

a) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1 - 2x - x^2} - (1 + x)}{x}$;

b) $\displaystyle\lim_{x\to 0} \frac{\sqrt[3]{8 + 3x - x^2} - 2}{x + x^2}$;

c) $\lim\limits_{x \to 0} \dfrac{\sqrt[3]{1 + \dfrac{x}{3}} - \sqrt[4]{1 + \dfrac{x}{4}}}{1 - \sqrt{1 - \dfrac{x}{2}}}$;

d) $\lim\limits_{x \to 0} \dfrac{\sqrt{1 + x} - \sqrt{1 - x}}{\sqrt[3]{1 + x} - \sqrt[3]{1 - x}}$.

3. Prove that

$$\lim_{x \to 0} \frac{\sin(\sin x)}{x} = 1.$$

4. Find the limit of

$$f(x) = \left(\frac{1 + x}{2 + x}\right)^{(1 - \sqrt{x})/(1 - x)}$$

as a) $x \to 0$; b) $x \to 1$; c) $x \to +\infty$.

20. Asymptotes

The graph of a bounded continuous function $y = f(x)$ defined in a finite interval can be constructed by plotting a finite number of points (x, y) and then joining them by a curve. Things are more complicated if $f(x)$ becomes infinite at certain points or if $f(x)$ is defined in an infinite interval (i.e., if the argument x becomes infinite). In this case we say that the function $f(x)$ or its graph has "infinite branches." However, the case of the straight line $y = kx + b$ shows that we can still get a perfectly satisfactory picture of the graph of a function even if the function has infinite branches. In fact, this example suggests comparing infinite branches with straight lines. The situation is particularly simple when an infinite branch approaches a straight line as the argument approaches infinity or certain other "exceptional points." In this case, the straight line in question is called an *asymptote* of the function $f(x)$ and the curve $y = f(x)$ is said to approach the straight line *asymptotically.* A function can approach its asymptote from one side only as in Figure 23, and it may even intersect its asymptote infinitely often as in Figure 24. Thus, to construct the graph of a function with an asymptote, it is important to analyze in detail the behavior of the function near its asymptote. In studying asymptotes, we distinguish three

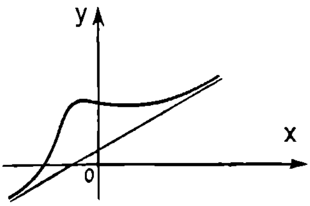

Fig. 23

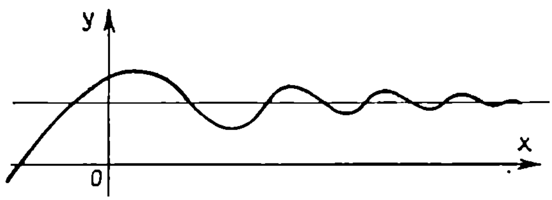

Fig. 24

cases, i.e.; *horizontal asymptotes* (parallel to the x-axis), vertical *asymptotes* (perpendicular to the x-axis) and *inclined asymptotes* (making an angle θ with the x-axis, where θ is not an integral multiple of $\pi/2$).

Case 1. Horizontal asymptotes

Suppose the horizontal line $y = a$ is an asymptote of the curve $y = f(x)$. Then the distance from the point $(x, f(x))$ to the line $y = a$ approaches zero as $x \to \infty$ from one side or another. But this distance is just $|f(x) - a|$, and hence

$$a = \lim_{x \to -\infty} f(x) \quad \text{or} \quad a = \lim_{x \to +\infty} f(x).$$

In other words, to find horizontal asymptotes, one must study the one-sided limits of $f(x)$ at infinity.

Example 1. Find the asymptotes and draw the graph of the function

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

Solution. Since $f(x)$ is bounded, it can have no vertical or inclined asymptotes. Moreover, since

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0,$$

only the x-axis can be a horizontal asymptote of $f(x)$. The function intersects this asymptote infinitely often, in fact at the points $x = k\pi$ ($k = \pm 1, \pm 2, \ldots$). Finally, $f(x)$ is even and hence symmetric with respect to the y-axis. This behavior is shown in Figure 25.

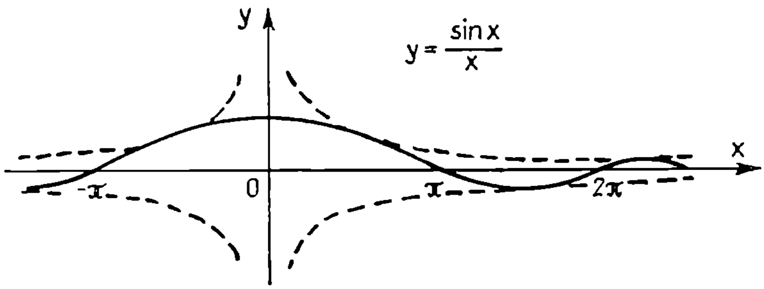

Fig. 25

Example 2. Find the asymptotes and draw the graph of the function

$$f(x) = \frac{x^2 + 2}{x^2 + 1}.$$

Solution. For the same reason as in Example 1, $f(x)$ has no vertical or inclined asymptotes, and moreover the line $y = 1$ is the only horizontal asymptote of $f(x)$ since

$$\lim_{x\to\infty} \frac{x^2 + 2}{x^2 + 1} = 1.$$

It follows from the formula

$$\frac{x^2 + 2}{x^2 + 1} = 1 + \frac{1}{x^2 + 1}$$

that the curve $y = f(x)$ lies above its asymptote. Finally, being even, $f(x)$ is symmetric with respect to the y-axis. This behavior is shown in Figure 26.

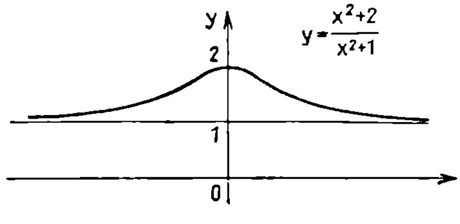

Fig. 26

Case 2. Vertical asymptotes

Suppose the vertical line $x = a$ is an asymptote of the curve $y = f(x)$. Then the distance from the point $(x, f(x))$ to the line $x = a$, i.e., the quantity $|x - a|$, approaches zero as x approaches a from one side or the other. At the same time, $f(x)$ itself must become infinite, since an asymptote is defined only for an infinite branch, and hence

$$\lim_{x\to a-} f(x) = \infty \quad \text{or} \quad \lim_{x\to a+} f(x) = \infty.$$

Thus, in looking for a vertical asymptote, we can confine our attention to points where $f(x)$ becomes infinite (and in particular is undefined).

Example 3. Find the horizontal and vertical asymptotes of the function

$$f(x) = e^{1/x},$$

and draw its graph.

Solution. Since

$$\lim_{x\to\infty} e^{1/x} = e^0 = 1,$$

the line $y = 1$ is a two-sided horizontal asymptote (i.e., an asymptote at both $-\infty$ and $+\infty$). The curve $y = e^{1/x}$ lies above its asymptote to the right of the origin, since $e^{1/x} > 1$ for $x > 0$, and below its asymptote to the left of the origin, since $e^{1/x} < 1$

for $x < 0$. The only point at which $f(x)$ fails to exist is at the origin, where we have

$$\lim_{x \to 0-} e^{1/x} = 0, \quad \lim_{x \to 0+} e^{1/x} = +\infty.$$

Therefore the y-axis is a vertical asymptote of $f(x)$, and the curve approaches this asymptote from the right, as shown in Figure 27.

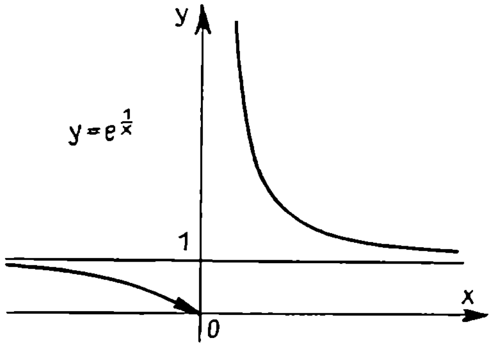

Fig. 27

Case 3. Inclined asymptotes

Finally suppose the inclined line $y = kx + b$ ($k \neq 0$) is an asymptote of the curve $y = f(x)$. Then the distance from the point $(x, f(x))$ to the line $y = ax + b$ approaches zero as x approaches infinity from one side or another, say as $x \to +\infty$. But it will be recalled from analytic geometry that this distance is just

$$\frac{|f(x) - kx - b|}{\sqrt{1 + k^2}},$$

and hence

$$\lim_{x \to +\infty} [f(x) - kx - b] = 0. \tag{3.13}$$

Using (3.13), we can determine the numbers k and b. To find k, we divide the function on the left by x and then pass to the limit, obtaining

$$\lim_{x \to +\infty} \left[\frac{f(x)}{x} - k - \frac{b}{x}\right] = \lim_{x \to +\infty} \frac{f(x)}{x} - k = 0,$$

i.e.,

$$k = \lim_{x \to +\infty} \frac{f(x)}{x}. \tag{3.14}$$

Once k is known, b is given by the formula

$$b = \lim_{x \to +\infty} [f(x) - kx]. \tag{3.15}$$

Example 4. Find the asymptotes and draw the graph of the function

$$f(x) = \frac{x^3}{2x^2 + 1}.$$

Solution. This function has no vertical asymptotes, since it is finite for all finite x, and no horizontal asymptotes, since it goes to infinity as $x \to \infty$. However, it has an inclined asymptote which can be found by using formulas (3.14) and (3.15):

$$k = \lim_{x \to +\infty} \frac{x^3}{x(2x^2 + 1)} = \frac{1}{2},$$

$$b = \lim_{x \to +\infty} \left[\frac{x^3}{2x^2 + 1} - \frac{x}{2}\right] = \lim_{x \to +\infty} \frac{-x}{2(2x^2 + 1)} = 0.$$

Therefore the line $y = x/2$ is an asymptote of $f(x)$ at $+\infty$. Since

$$\frac{x^3}{2x^2 + 1} = \frac{x}{2} - \frac{x}{4x^2 + 2},$$

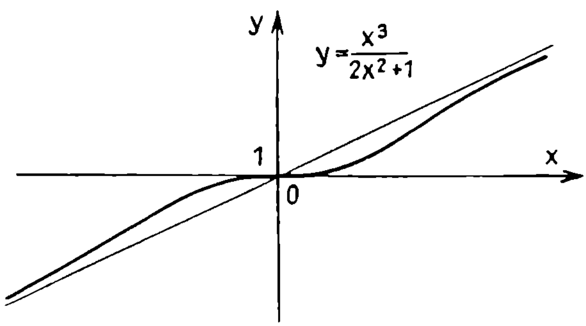

Fig. 28

the curve $y = f(x)$ lies below this asymptote if $x > 0$. Since $f(x)$ is odd, its graph is symmetric with respect to the origin, as shown in Figure 28. In particular, the line $y = x/2$ is the only asymptote of $f(x)$.

PROBLEMS

1. Find the asymptotes and draw the graphs of the following functions:

a) $y = \dfrac{x-4}{2x+4}$; b) $y = \dfrac{x^2}{2-2x}$;

c) $y = \dfrac{x^2}{x^2-4}$; d) $y = \dfrac{x^3}{1-x^2}$.

2. Find the asymptotes of the following functions:

a) $y = \sqrt{x^2+1} - \sqrt{x^2-1}$; b) $y = \sqrt{x^2+1} + \sqrt{x^2-1}$;

c) $y = x - \dfrac{1}{\sqrt{x}}$.

21. The Modulus of Continuity. Uniform Continuity

DEFINITION 1. *Given a function $f(x)$ continuous in an interval I,*[10] *by the modulus of continuity of $f(x)$ in I is meant the quantity*

$$\omega(\delta) = \sup_{\substack{x,\, y \in I \\ |x-y| \leqslant \delta}} |f(x) - f(y)|.$$

THEOREM 3.14. *Let $\omega(\delta)$ be the modulus of continuity of a function continuous in an interval I. Then $\omega(\delta)$ is increasing and moreover*

$$\lambda = \lim_{\delta \to 0+} \omega(\delta) \tag{3.16}$$

exists.

10. Here I may be open, closed or half-open.

Proof. Increasing δ cannot decrease the set of numbers whose least upper bound is being taken. But this cannot decrease $\omega(\delta)$, i.e., $\delta' > \delta$ implies $\omega(\delta') \geqslant \omega(\delta)$. The existence of (3.16) follows from Theorem 2.20 and the fact that $\omega(\delta)$ is bounded and monotonic in the set $x, y \in I$, $|x - y| \leqslant \delta$ (recall Theorem 3.9).

The case $\lambda = 0$ is of greatest interest. If $\lambda = 0$, then given any $\varepsilon > 0$, there is a number $\delta_0 = \delta_0(\varepsilon) > 0$ such that $\omega(\delta) < \varepsilon$ for all $\delta < \delta_0$. In other words, if $\lambda = 0$, then given any $\varepsilon > 0$, therc is a $\delta_0 = \delta(\varepsilon)$ such that

$$|f(x) - f(y)| < \varepsilon \tag{3.17}$$

for any two points $x, y \in I$ whenever $|x - y| < \delta_0$. This property does *not* follow from the continuity of $f(x)$ in I. In fact, all that is implied by the continuity of $f(x)$ in I is that given any $\varepsilon > 0$ and $x \in I$, there is a $\delta_0 = \delta_0(\varepsilon, x)$ such that (3.17) holds for any $y \in I$ whenever $|x - y| < \delta_0$. Here δ_0 in general depends on x, since all we know is that $f(x)$ is continuous in I, i.e., at every point $x \in I$. If there were only finitely many points $x \in I$, we could take δ_0 to be the smallest of the positive numbers $\delta_0(\varepsilon, x)$, $x \in I$, and then (3.17) would hold for any two points $x, y \in I$ whenever $|x - y| < \delta_0$. But I contains infinitely many points, being an interval. This suggests defining

$$\delta_0 = \inf_{x \in I} \delta_0(\varepsilon, x),$$

and then asserting that (3.17) holds for any two points $x, y \in I$ whenever $|x - y| < \delta_0$. However, this statement becomes meaningless if δ_0 turns out to be zero!

The above considerations motivate

DEFINITION 2. *Let $f(x)$ be continuous in an interval I, with modulus of continuity $\omega(\delta)$. Then $f(x)$ is said to be uniformly continuous in I if*

$$\lambda = \lim_{\delta \to 0+} \omega(\delta) = 0.$$

Remark. In other words, if $f(x)$ is uniformly continuous in I, then given any $\varepsilon > 0$, there is a $\delta_0 = \delta_0(\varepsilon)$ *independent of* x such that (3.17) holds for any two points $x, y \in I$ whenever $|x - y| < \delta_0$.

Example. The function

$$f(x) = \cos\frac{\pi}{x} \quad \text{if} \quad x \in (0, 1)$$

is continuous but not uniformly continuous in the open interval (0,1). In fact, let

$$x_n = \frac{1}{n}, \quad y_n = \frac{1}{n+1}, \quad \delta_n = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}.$$

Then $\delta_n \to 0$ as $n \to \infty$, but

$$|f(x_n) - f(y_n)| = |\cos n\pi - \cos(n+1)\pi| = 2$$

and hence

$$\omega(\delta_n) = \sup_{\substack{x, y \in (0,1) \\ |x-y| \leqslant \delta_n}} |f(x) - f(y)| \geqslant 2,$$

which implies

$$\lambda = \lim_{\delta \to 0+} \geqslant 2.$$

Remarkably enough, it turns out that continuity implies uniform continuity if the interval is *closed:*

THEOREM 3.15. *If $f(x)$ is continuous in a closed interval $[a, b]$, then $f(x)$ is uniformly continuous in $[a, b]$.*

Proof. Let $\omega(\delta)$ be the modulus of continuity of $f(x)$ in $[a, b]$, and let

$$\lambda = \lim_{\delta \to 0+} \omega(\delta).$$

Then we must prove that $\lambda = 0$. Suppose to the contrary that $\lambda > 0$. Then

$$\omega(\delta) \geqslant \lambda > 0$$

for all δ. Therefore, by the definition of $\omega(\delta)$, given any positive integer n, there are points $x_n, y_n \in [a, b]$ such that $|x_n - y_n| < 1/n$ and

$$|f(x_n) - f(y_n)| \geqslant \frac{\lambda}{2}. \tag{3.18}$$

Since $x_n \in [a, b]$ for all n, the sequence of points $\{x_n\}$ is bounded. Hence by Theorem 2.18, Corollary 1, $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ converging to a point $x_0 \in [a, b]$. Clearly, $\{y_{n_k}\}$ also converges to x_0, since

$$y_{n_k} - y_0 = (y_{n_k} - x_{n_k}) + (x_{n_k} - x_0) \to 0 + 0 = 0$$

as $k \to \infty$. From this and the continuity of $f(x)$ at x_0 we deduce that

$$\lim_{k \to \infty} [f(x_{n_k}) - f(y_{n_k})] = f(x_0) - f(x_0) = 0.$$

In other words,

$$|f(x_{n_k}) - f(y_{n_k})| < \frac{\lambda}{2}$$

for sufficiently large k, contrary to (3.18). This contradiction proves the theorem.

PROBLEMS

1. Prove that $\lambda = 2$ in the example on p. 113.

2. Prove that if $\omega(\delta)$ is the modulus of continuity of the function $f(x) = x^3$ in the interval $[0, 1]$, then $\omega(\delta) \leqslant 3\delta$.

3. Prove that the unbounded function

$$f(x) = x + \sin x$$

is uniformly continuous on the whole real line.

4. Is the function $f(x) = x^2$ uniformly continuous

a) In the interval $[-a, a]$ where $a > 0$ is arbitrarily large;

b) In the infinite interval $(-\infty, +\infty)$?

5. Prove that the function

$$f(x) = \frac{|\sin x|}{x}$$

is uniformly continuous in each of the intervals $(-1, 0)$ and $(0, +1)$ but not in their union.

6. Which of the following functions are uniformly continuous in the indicated intervals:

a) $f(x) = \dfrac{x}{4 - x^2}$ $(-1 \leqslant x \leqslant 1)$;

b) $f(x) = \ln x$ $(0 < x < 1)$;

c) $f(x) = \dfrac{\sin x}{x}$ $(0 < x < \pi)$;

d) $f(x) = e^x \cos \dfrac{1}{x}$ $(0 < x < 1)$;

e) $f(x) = \arctan x$ $(-\infty < x < +\infty)$;

f) $f(x) = \sqrt{x}$ $(1 \leqslant x < +\infty)$;

g) $f(x) = x \sin x$ $(0 \leqslant x < +\infty)$?

Answers to Even-Numbered Problems

Sec. 1, p. 8

2. $\varphi(1/\sqrt{3})$ and $\varphi(1)$ do not exist; $\varphi(1/\sqrt{2})$ exists and equals $2\sqrt{2} - 2$.

4. a) $1, 4, 9, 16, 25, \ldots$; b) $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}, \ldots$;

c) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$; d) $2, 4, 8, 16, 32, \ldots$

6. If x is rational, then $D(x) = 1$ and hence $D^n(x) = 1$, while if x is irrational, then $D(x) = 0$ and hence $D^n(x) = 0$. Therefore $D^n(x) = D(x)$ for all x.

Sec. 2, p. 11

2. $x \leqslant -1$; $x \geqslant 2$.

4. a) $-1.01 < x < -0.99$; b) $x \leqslant -8$; $x \geqslant 12$;

c) $x < -\frac{1}{2}$; d) $0 < x < \frac{2}{3}$; e) $-6 \leqslant x \leqslant 6$;

f) $x > -\frac{1}{2}$; g) $-\frac{1}{2} < x < \frac{1}{2}$.

6. If $x > 0$ then $|x| = x$, $\operatorname{sgn} x = 1$, and hence $|x| = x = x \operatorname{sgn} x$. If $x = 0$ then $|x| = 0$, $\operatorname{sgn} 0 = 0$ and hence $|x| = 0 = x \operatorname{sgn} x$. If $x < 0$ then $|x| = -x$, $\operatorname{sgn} x = -1$ and hence $|x| = -x = x \operatorname{sgn} x$.

Sec. 3, p. 13

2. 2.11, −0.25, −1.86.

4. The conversion from degrees Centigrade to degrees Fahrenheit; $x = 40$, $y = 176$; $y = \frac{9}{5}x + 32$, $x = \frac{5}{9}y - 32$.

Sec. 4, p. 19

2. a), b), i) and l) are even, c), g), k) and m) are odd, d,) e), f), h) and j) are neither even nor odd.

4. a), d), e) ,g), h) and i) are periodic. The smallest positive period of a) and d) is π, that of e) is 2 and that of h) and i) is 2π. The function g) has any nonzero number as a period, and hence has no smallest positive period.

6. a) Strictly increasing in $(-\infty, +\infty)$; b) Strictly decreasing in $(-\infty, 0)$, strictly increasing in $(0, +\infty)$; c) Strictly decreasing in $(-\infty, -1)$, strictly increasing in $(-1, +\infty)$; d) Strictly increasing in $(-\infty, +\infty)$; e) Strictly increasing in $(-\infty, +\infty)$; f) Strictly decreasing in $(-\infty, 0)$, strictly increasing in $(0, +\infty)$; g) Strictly decreasing in $(-\infty, 0)$, nondecreasing in $(0, +\infty)$; h) Strictly increasing in the intervals $\left(-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi\right)$, strictly decreasing in the intervals $\left(\frac{\pi}{2} + 2k\pi, \frac{3}{2}\pi + 2k\pi\right)$ where k is any nonnegative integer.

Sec. 5, p. 23

2. a) If $x = 0.417417\ldots$, then $1000x = 417.417417\cdots$ and hence $1000x - x = 417.417417\cdots - 0.417417\cdots = 417$

or

$$999x = 417,$$

i.e.,

$$x = \frac{417}{999} = \frac{139}{333};$$

b) $\frac{2329}{999}$; c) $\frac{23}{25}$.

Sec. 6, p. 31

2. Use Theorem 2.3.

4. Only a) and e) are bounded.

6. $y = \tan x$.

Sec. 7, p. 36

2. a) 1; b) $\frac{1}{3}$; c) $\sin 1$; d) $\frac{m}{n}$; e) 0; f) $\frac{1}{64}$; g) 0.

Sec. 8, p. 42

2. The left-hand limits are all 1, the right-hand limits are all 0.

4. Both limits equal 3.

6. The left-hand limit is 0, the right-hand limit does not exist.

Sec. 9, p. 48

2. No, since $f(x) = 0$ at every point

$$x = \frac{1}{\dfrac{\pi}{2} + 2k\pi} \quad (k = 0, \pm 1, \pm 2, \ldots).$$

4. a) $\dfrac{1}{2}$; b) $\dfrac{1}{2}$; c) $\dfrac{1}{4}$; d) $\dfrac{2}{\pi}$; e) -1.

Sec. 10, p. 52

2. a) $\dfrac{1}{2}$; b) $\dfrac{1}{5^5}$; c) $\left(\dfrac{3}{2}\right)^{30}$.

Sec. 11, p. 57

2. a) $\frac{2}{3}$; b) 0; c) $-\frac{1}{2}$; d) $\frac{3}{4}$; e) 0; f) $\frac{1}{2}$.

4. $\max E = \sup E = 1$, $\min E$ does not exist, $\inf E = 0$.

Sec. 12, p. 61

2. Let $a = 0$, $b = 2$ in Prob. 1.

4. $$|f(x') - f(x'')| \leqslant \left|x' \sin\frac{1}{x'}\right| + \left|x'' \sin\frac{1}{x''}\right| \leqslant |x'| + |x''|,$$

and hence $|f(x') - f(x'')| < \varepsilon$ if $|x'| < \dfrac{\varepsilon}{2}$, $|x''| < \dfrac{\varepsilon}{2}$; on the other hand, in any neighborhood of $x = 0$ there are points of the form

$$x' = \frac{1}{2k\pi}, \quad x'' = \frac{1}{(2k+1)\pi},$$

and hence given any $0 < \varepsilon < 2$,

$$|g(x') - g(x'')| = \left|\cos\frac{1}{x'} - \cos\frac{1}{x''}\right| = 2 > \varepsilon.$$

Sec. 13, p. 71

2. 0 if $|a| < 1$, 0 if $|a| > 1$, $\frac{1}{2}$ if $a = 1$; the sequence is divergent if $a = -1$.

4. $\dfrac{1}{e}$.

6. It follows from Prob. 5 and the proof of Theorem 2.23 (together with the definition of e) that

$$\left(1 + \frac{1}{n}\right)^{n+1} > e > \left(1 + \frac{1}{n}\right)^n.$$

Therefore

$$(n + 1)\ln\left(1 + \frac{1}{n}\right) > 1 > n\ln\left(1 + \frac{1}{n}\right),$$

which implies

$$\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}.$$

Here we anticipate the fact that $\log_a x$ is strictly increasing if $a > 1$ (see p. 94).

Sec. 14, p. 78

2. No. It means that $f(x)$ is bounded in a neighborhood of x_0.

4. a) and d).

6. a) $f[g(x)]$ is continuous; $g[f(x)]$ is discontinuous at $x = 0$; b) $f[g(x)]$ is discontinuous at $x = -1, 0, 1$; $g[f(x)]$ is continuous; c) $f[g(x)]$ and $g[f(x)]$ are continuous.

Sec. 15, p. 82

2. a) A discontinuity of the second kind at $x = -1$; b) A removable discontinuity at $x = -1$; c) Discontinuities of the second kind at $x = -2$ and $x = 1$; d) A removable discontinuity at $x = -1$; e) A discontinuity of the second kind at $x = 0$; f) Removable discontinuities at $x = \pm 2$; g) A discontinuity of the second kind at $x = 0$; h) Discontinuities of the first kind at $x = \frac{1}{k}$ $(k = \pm 1, \pm 2, \ldots)$ and a discontinuity of the second kind at $x = 0$; i) A discontinuity of the first kind at $x = 1$ and a discontinuity of the second kind at $x = 0$.

Sec. 16, p. 88

2. If $f(x) = \sin x - x + 1$, then

$$f\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2} - \frac{2\pi}{3} + 1 > 0,$$

$$f\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{3\pi}{4} + 1 < 0.$$

Now apply Theorem 3.6.

4. Every neighborhood of $x = 0$ contains an interval of the form

$$I_k = \left[\frac{1}{(2k+1)\pi}, \frac{1}{2k\pi}\right].$$

But $f(x)$ is continuous in I_k and

$$f\left(\frac{1}{(2k+1)\pi}\right) = -1, \quad f\left(\frac{1}{2k\pi}\right) = +1.$$

Therefore $f(x)$ takes every value between -1 and $+1$ in I_k, by Theorem 3.7.

6. Let

$$f(x) = \begin{cases} x + 1 & \text{if } -1 \leqslant x \leqslant 0, \\ -x & \text{if } \quad 0 < x \leqslant 1. \end{cases}$$

Sec. 17, p. 91

2. If $y = \dfrac{1-x}{1+x}$, then $x = \dfrac{1-y}{1+y}$.

4. $x = y - k$ if $2k \leqslant y < 2k + 1$ $(k = 0, \pm 1, \pm 2, \ldots)$.

Sec. 18, p. 100

2. a) $-1 \leqslant x \leqslant 1$; b) $0 \leqslant x \leqslant 1$; c) $0 \leqslant x \leqslant 1$;
d) $-1 \leqslant x \leqslant 0$.

4. Solving the equation

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

or

$$e^{2y} - 2xe^y - 1 = 0,$$

we obtain

$$e^y = x \pm \sqrt{x^2 + 1}$$

or

$$y = \operatorname{arc\,sinh} x = \ln (x \pm \sqrt{x^2 + 1}),$$

where the plus sign must be chosen so that the argument of the logarithm will be positive. Similarly

$$\operatorname{arc\,cosh} x = \ln (x + \sqrt{x^2 - 1}),$$

provided we choose the branch which is nonnegative for $-1 \leqslant x < +\infty$.

Solving the equation

$$x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

or

$$e^{2y} = \frac{1 + x}{1 - x},$$

we obtain

$$y = \frac{1}{2} \ln \frac{1 + x}{1 - x}.$$

Sec. 19, p. 104

2. a) -2; b) $\frac{1}{4}$; c) $\frac{7}{36}$; d) $\frac{3}{2}$.

4. a) $\frac{1}{2}$; b) $\sqrt{\frac{2}{3}}$; c) 1.

Sec. 20, p. 111

2. a) $y = 0$; b) $y = \pm 2x$; c) $x = 0, y = x$.

Sec. 21, p. 114

2. If $x, y \in (0,1)$, then

$$\begin{aligned}|x^3 - y^3| &= |(x^2 + xy + y^2)(x - y)| \\ &= (x^2 + xy + y^2)\,|x - y| \leqslant 3\,|x - y|.\end{aligned}$$

But then $|x - y| \leqslant \delta$ implies $|x^3 - y^3| \leqslant 3\delta$, and hence

$$\omega(\delta) = \sup_{\substack{x,y \in (0,1) \\ |x-y| \leqslant \delta}} |x^3 - y^3| \leqslant 3\delta.$$

4. a) Yes; b) No.

6. a), c), e) and f).

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